

## ON TOPOLOGICAL 1D GRAVITY. I

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ABSTRACT. In topological 1D gravity, the genus zero one-point function combined with the gradient of the action function leads to a spectral curve and its special deformation. After quantization, the partition function is identified as an element in the bosonic Fock space uniquely specified by the Virasoro constraints.

## 1. INTRODUCTION

This is the first part of a series of papers in which we will systematically study the topological 1D gravity, in the framework of emergent geometry and quantum deformation theory of its spectral curve.

Our motivation to study topological 1D gravity was originally to gain some more understanding of topological 2D gravity. This seems to be also the motivations of earlier works on this subject [8, 9, 2, 10, 12]. As it turns out, not only does topological 1D gravity share similar properties as topological 2D gravity, such as their connections with integrable hierarchies and their Virasoro constraints, but also the derivations of these properties are much simpler. Furthermore, it admits some easy generalizations and some connections with other topics to be reported in later parts of this series that makes it have some independent interests. This series is a companion series to a series on related work on topological 2D gravity whose first part is [18].

More than twenty years ago, topological 2D gravity was studied intensively from the point of view of double scaling limits of large  $N$  random matrices [4, 5, 6]. A remarkable connection with the intersection theory of moduli spaces of algebraic curves was made by Witten [15]. In his proof of Witten Conjecture, Kontsevich [7] introduced a different kind of matrix models. Around the same time, topological 1D gravity arose in the context of double scaling limits of large  $N$   $O(N)$  vector models in the references [8, 2, 10, 12] mentioned above. They were originally called (branching) polymer models or (branching) chain models. In [9], it was proposed that polymer model is equivalent to a topological theory of 1D gravity. The author gets interested in this theory because it provides another example in which one can study mirror

symmetry from the point of view of emergent geometry and quantum deformation theory.

In [18] we have proposed to develop quantum deformation theory as an approach to mirror symmetry. By this we mean the genus zero free energy on the big phase space (the space of coupling constants of all gravitational descendants) leads to a geometric structure and its special deformations, and by quantization of this picture one gets constraints the free energy in all genera which suffice to determine the whole partition function. We will refer to this as the emergent geometry. Or more precisely, by emergent geometry we mean the geometric picture better seen or understood when one goes to the big phase space. In [18], we have shown that the emergent geometry of topological 2D gravity is the quantum deformation of theory of the Airy curve:

$$y = \frac{1}{2}x^2.$$

For related results, see [13, 3]. One of the main results of this paper is that the emergent geometry of topological 2D gravity is the quantum deformation theory of the signed Catalan curve:

$$y = -\frac{1}{\sqrt{2}}z + \frac{\sqrt{2}}{z}.$$

In this paper we will also study the following coordinate change on the big phase space:

$$\begin{aligned} I_0 &= \sum_{k=1}^{\infty} \frac{1}{k} \sum_{p_1+\dots+p_k=k-1} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_k}}{p_k!}, \\ I_k &= \sum_{n \geq 0} t_{n+k} \frac{I_0^n}{n!}, \quad k \geq 1. \end{aligned}$$

These series were introduced in [14] to express the free energy in the context of topological 2D gravity. By understanding them as new coordinates on the big phase space, one can gain better understanding of the global nature of the behavior of the theory on the big phase space. For example, we use two different methods to show that for topological

1D gravity,

$$\begin{aligned}
F_0 &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1}, \\
F_1 &= \frac{1}{2} \ln \frac{1}{1 - I_1}, \\
F_g &= \sum_{\sum_{j=2}^{2g-1} \frac{j-1}{2} l_j = g-1} \langle \tau_2^{l_2} \cdots \tau_{2g-1}^{l_{2g-1}} \rangle_g \prod_{j=2}^{2g-1} \frac{1}{l_j!} \left( \frac{I_j}{(1 - I_1)^{(j+1)/2}} \right)^{l_j}, \quad g \geq 2.
\end{aligned}$$

Partly fulfilling our wish to gain more insights on topological 2D gravity by studying topological 1D gravity, we will in a sequel to [18] use one of these methods to prove that similar formulas hold for topological 2D gravity:

$$\begin{aligned}
F_0^{2D} &= (1 - I_1)^2 \frac{I_0^3}{3!} + (1 - I_1) I_2 \frac{I_0^4}{4!} + I_2^2 \frac{I_0^5}{5!} \\
&+ \sum_{n \geq 6} (-1)^{n-1} \left[ (n-5)(1 - I_1) I_{n-2} \right. \\
&- \left. \frac{1}{2} \sum_{j=2}^{n-3} \left( \binom{n-3}{j} - 2 \binom{n-3}{j-1} + \binom{n-3}{j-2} \right) I_j I_{n-1-j} \right] \frac{I_0^n}{n!}, \\
F_1^{2D} &= \frac{1}{24} \ln \frac{1}{1 - I_1}, \\
F_g^{2D} &= \sum_{\sum_{j=2}^{3g-1} \frac{j-1}{3} l_j = g-1} \langle \tau_2^{l_2} \cdots \tau_{3g-2}^{l_{3g-2}} \rangle_g \prod_{j=2}^{3g-2} \frac{1}{l_j!} \left( \frac{I_j}{(1 - I_1)^{(2j+1)/3}} \right)^{l_j},
\end{aligned}$$

where  $g > 1$ . In this case the formula for  $F_1$  was due to [11], the formula for  $F_g$  ( $g > 1$ ) was conjectured in [14], and the formula for  $F_0$  seems to be new. Such formulas indicate that one can study the behavior of free energy of topological 1D and 2D gravity near  $t_1 = 1$ . This is where one gets multicritical phenomenon originally studied in the matrix model or vector model approach [8, 2, 10]. Note we prove in this paper for topological 1D gravity:

$$F = \frac{1}{2} \log(1 - t_1) + \sum_{\substack{g, n \geq 0 \\ 2g-2+n > 0}} \sum_{\substack{a_2, \dots, a_n \neq 1 \\ \sum a_j = 2g-2+n}} \frac{\langle \prod_{j=1}^n \tau_{a_j} \rangle_g}{(1 - t_1)^{g-1+n}} \prod_{j=1}^n t_{a_j},$$

and in topological 2D gravity one have similar formula by dilaton equation, for example,

$$(1) \quad F_0(\mathbf{t}) = \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)(1-t_1)^k} \sum_{\substack{p_1+\dots+p_{k+2}=k-1 \\ p_j \neq 1, \quad j=1,\dots,k+2}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_{k+2}}}{p_{k+2}!}.$$

By looking at such formulas one might get the wrong impression that it is impossible to consider the theories at  $t_1 = 1$ . In later parts of this series and a sequel to [16], we will address these issues. Another result proved in this paper is the following analogue of Kontsevich's main identity [7]:

$$\sum \langle \tau_0^{m_0} \dots \tau_n^{m_n} \rangle_g \prod_{j=0}^n \frac{t_j^{m_j}}{m_j!} = \sum_{\Gamma \in \mathcal{G}^c} \frac{1}{|\text{Aut}(\Gamma)|} \prod_{v \in V(\Gamma)} \lambda^{\text{val}(v)-2} t_{\text{val}(v)-1}.$$

In this part of the series, we will understand the topological 1D gravity as the  $N = 1$  case of matrix models for topological gravity in the physics literature. More precisely, we will work with a formal Gaussian integral, with infinitely many parameters giving the coupling constants to gravitational descendants. This is the mean field theory of the topological 1D gravity. This point of view of connecting topological 1D gravity and 2D gravity makes it possible to generalize to topological 1D gravity coupled with topological matters, a topic to be discussed in a later part of this series. The problem of building the theory on integrations over moduli spaces of some geometric objects will be addressed also in a subsequent part of this series.

As the companion work in topological 2D gravity [16, 17], we are inspired by [1]. In this series we elaborate on an example not included in their beautiful work.

Let us now sketch the contents of the rest of this paper. In §2 we explain how the simple idea of completing the squares, when used repeatedly, leads to renormalization of coupling constants in topological 1D gravity. Furthermore, this process can be embedded in the formal gradient flow and the limit point is the single critical point of the action function in the formal setting. The  $I$ -coordinates naturally appear as the Taylor coefficients at the critical point.

In §3 we reformulate the formula for  $I$ -coordinates in  $t$ -coordinates in terms of Feynman rules. This raises the problem of realizing them by some quantum field theory. We propose the solution in §4 as the mean field theory of topological 1D gravity. We define and develop this theory based on formal Gaussian integrals and their properties

in this Section. In particular, the formulas of the free energy in  $I$ -coordinates will be proved based on translation invariance of formal Gaussian integrals. This is a mathematically rigorous approach to the saddle point method in this setting.

We study the applications of flow equations and polymer equation for topological 1D gravity [8] in §5. They are the analogues of the KdV hierarchy and the string equation in topological 2D gravity respectively. In this Section we present two more derivations of the formula for  $F_0$  in  $I$ -coordinates.

Some applications of Virasoro constraints in topological 1D gravity [8] will be presented in §6. These include our fourth derivation of formula for  $F_0$  in  $I$ -coordinates and our second derivation of the formula for  $F_g$  ( $g \geq 1$ ).

In §7 we rederive Virasoro constraints from the point of view of operator algebras. We present W-constraints and another version of Virasoro constraints for topological 1D gravity. As it turns out, this second version of Virasoro constraints is in closer analogy with the Virasoro constraints for topological 2D gravity and it is the version we need to develop the quantum deformation in a later Section.

We define and compute two kinds of  $n$ -point functions in topological gravity in §8. The computations rely heavily on the flow equations. We also obtain some recursion relation for  $n$ -point functions. We derive some Feynman rules for  $n$ -point functions in §9. These rules express the  $n$ -point functions in terms of genus zero free energy restricted to the small phase space. In topological 2D gravity similar rules were proposed in [11].

In §10 we show that the genus zero one-point function combined with the gradient of the action function leads us to the spectral curve and its special deformation for topological 1D gravity. We also establish the uniqueness of the special deformation. After quantizing the special deformation of the spectral curve in §11, we identify the partition function as an element in the bosonic Fock space uniquely specified by the Virasoro constraints in §7.

We summarize our results in the concluding §12.

## 2. RENORMALIZATION OF THE ACTION FUNCTION

In this and the next Sections, we will study the  $I$ -coordinates from various points of views.

### 2.1. The effective action function of the topological 1D gravity.

It is the following formal power series in  $x$  depending on infinitely many

parameters  $t_0, \dots, t_n, \dots$ :

$$(2) \quad S = -\frac{1}{2}x^2 + \sum_{n \geq 1} t_{n-1} \frac{x^n}{n!},$$

The coefficients  $t_n$ 's will be called the *coupling constants*. Since we do not concern ourselves with the issue of the convergence of the above series, we will treat the  $t_n$ 's either as formal variables, or take a truncation

$$(3) \quad t_{n+1} = t_{n+2} = \dots = 0$$

for suitable  $n$ .

**2.2. The dialton shift.** One can rewrite  $S$  more uniformly as follows:

$$(4) \quad S = \sum_{n \geq 0} \tilde{t}_n \frac{x^{n+1}}{(n+1)!},$$

where  $\tilde{t}_n = t_n - \delta_{n,1}$ . This shift in coordinates is called the *dilaton shift*.

**2.3. Space of action functions.** When we regard  $t_n$ 's as formal variables, we will consider the space  $\mathcal{S}$  consisting of formal power series of the form

$$(5) \quad S = -\frac{1}{2}x^2 + \sum_{n \geq -1} T_n \frac{x^{n+1}}{(n+1)!},$$

where each  $\tilde{T}_n$  is a formal power series in  $t_0, t_1, \dots$ ; we require furthermore that

$$(6) \quad T_1|_{t_0=t_1=\dots=0} = 0,$$

i.e., the constant term of  $\tilde{T}_1$  is 0.

When we take the truncation (3),  $\tilde{t}_0, \dots, \tilde{t}_n$  give us coordinates on the  $n+1$ -dimensional Euclidean space  $\mathcal{S}_n$  of degree  $n+1$  polynomials without constant terms.

**2.4. Renormalization of the coupling constants by completing the squares.** Let us take

$$S = -\frac{1}{2}(1-t_1)x^2 + \sum_{n \geq 0} t_{n-1} \frac{x^n}{n!}$$

and apply the procedure of completing the square: let  $\tilde{x} = x - x_1$ , where  $x_1 = \frac{t_0}{1-t_1}$ , then

$$\begin{aligned}
S &= t_{-1} - \frac{1}{2}(1-t_1)x^2 + t_0x + \sum_{n \geq 3} t_{n-1} \frac{x^n}{n!} \\
&= t_{-1} + \frac{1}{2} \frac{t_0^2}{1-t_1} - \frac{1}{2}(1-t_1)\tilde{x}^2 + \sum_{n \geq 3} t_{n-1} \frac{(\tilde{x} + \frac{t_0}{1-t_1})^n}{n!} \\
&= t_{-1} + \frac{1}{2} \frac{t_0^2}{1-t_1} - \frac{1}{2}(1-t_1)\tilde{x}^2 + \sum_{n \geq 3} t_{n-1} \sum_{m=0}^n \frac{\tilde{x}^m}{m!} \frac{1}{(n-m)!} \left( \frac{t_0}{1-t_1} \right)^{n-m} \\
&= \left( t_{-1} + \frac{1}{2} \frac{t_0^2}{1-t_1} + \sum_{n \geq 3} \frac{t_{n-1}}{n!} \left( \frac{t_0}{1-t_1} \right)^n \right) + \tilde{x} \sum_{n \geq 3} \frac{t_{n-1}}{(n-1)!} \left( \frac{t_0}{1-t_1} \right)^{n-1} \\
&\quad - \frac{1}{2} \left( 1-t_1 - \sum_{n \geq 3} t_{n-1} \frac{1}{(n-2)!} \left( \frac{t_0}{1-t_1} \right)^{n-2} \right) \tilde{x}^2 \\
&\quad + \sum_{m=3}^{\infty} \frac{\tilde{x}^m}{m!} \sum_{n \geq m} t_{n-1} \frac{1}{(n-m)!} \left( \frac{t_0}{1-t_1} \right)^{n-m} \\
&= \left( t_{-1} + \frac{1}{2} \frac{t_0^2}{1-t_1} + \sum_{n \geq 3} \frac{t_{n-1}}{n!} \left( \frac{t_0}{1-t_1} \right)^n \right) + \tilde{x} \sum_{n \geq 2} \frac{t_n}{n!} \left( \frac{t_0}{1-t_1} \right)^n \\
&\quad - \frac{1}{2} \left( 1 - \sum_{n \geq 0} t_{n+1} \frac{1}{n!} \left( \frac{t_0}{1-t_1} \right)^n \right) \tilde{x}^2 + \sum_{m=3}^{\infty} \frac{\tilde{x}^m}{m!} \sum_{n \geq 0} t_{n+m-1} \frac{1}{n!} \left( \frac{t_0}{1-t_1} \right)^n.
\end{aligned}$$

From this computation we define the renormalization transformation  $R : \mathcal{S} \rightarrow \mathcal{S}$ :

$$(7) \quad (t_{-1}, t_0, t_1, \dots) \mapsto (\hat{t}_{-1}, \hat{t}_0, \hat{t}_1, \dots),$$

where

$$(8) \quad \hat{t}_{-1} = t_{-1} + \frac{1}{2} \frac{t_0^2}{1-t_1} + \sum_{n \geq 3} \frac{t_{n-1}}{n!} \left( \frac{t_0}{1-t_1} \right)^n,$$

$$(9) \quad \hat{t}_0 = \sum_{n \geq 2} \frac{t_n}{n!} \left( \frac{t_0}{1-t_1} \right)^n,$$

$$(10) \quad \hat{t}_1 = \sum_{n \geq 0} t_{n+1} \frac{1}{n!} \left( \frac{t_0}{1-t_1} \right)^n,$$

$$(11) \quad \hat{t}_m = \sum_{n \geq 0} t_{n+m} \frac{1}{n!} \left( \frac{t_0}{1-t_1} \right)^n, \quad m \geq 2.$$

**2.5. Geometric interpretation of the renormalization transformation.** The renormalization transformation is related to Newton's algorithm as follows. Formally, consider the tangent line to

$$y = \frac{\partial}{\partial x} S = -(1 - t_1)x + t_0 + \sum_{n \geq 2} t_n \frac{x^n}{n!}$$

at  $x_0 = 0$ . Because

$$\begin{aligned} y|_{x=0} &= t_0, \\ \frac{\partial}{\partial x} y|_{x=0} &= -(1 - t_1), \end{aligned}$$

so the tangent line is given by

$$y = -(1 - t_1)x + t_0.$$

This line intersection the  $x$ -axis at  $x_1 = \frac{t_0}{1-t_1}$ . Then  $R(t_{-1}, t_0, t_1, \dots)$  are the Taylor coefficients of  $S$  at  $x = x_1$ .

**2.6. Renormalization flow and the gradient flow.** Consider the gradient flow of  $S$ :

$$(12) \quad \frac{\partial x(s)}{\partial s} = \frac{\partial S}{\partial x} = -x(s) + \sum_{n \geq 0} t_n \frac{x(s)^n}{n!}$$

Let us first show that it can be formally solved by power series method. Write

$$x(s) = \sum_{n \geq 1} a_n s^n,$$

then the above equation gives:

$$\begin{aligned} & a_1 + 2a_2s + 3a_3s^2 + \dots \\ &= t_0 - \sum_{m \geq 1} a_m s^m + \sum_{l \geq 1} \frac{t_l}{l!} \left( \sum_{n=1}^{\infty} a_n t^n \right)^l \\ &= t_0 - \sum_{m \geq 1} a_m s^m + \sum_{l \geq 1} t_l \sum_{\substack{k_1 + \dots + k_r = l \\ k_1, \dots, k_r \geq 0}} \frac{a_1^{k_1}}{k_1!} \dots \frac{a_r^{k_r}}{k_r!} \cdot s^{\sum_{j=1}^r j k_j}. \end{aligned}$$

Therefore,

$$\begin{aligned} a_1 &= t_0, \\ (m+1)a_{m+1} &= -a_m + \sum_{\substack{\sum_{j=1}^r j k_j = m \\ k_1, \dots, k_r \geq 0}} \frac{a_1^{k_1}}{k_1!} \dots \frac{a_r^{k_r}}{k_r!} \cdot t_{\sum_{j=1}^r k_j}, \end{aligned}$$



for  $m \geq 1$ . Therefore, one can recursively find  $a_m$ . For example,

$$\begin{aligned} a_2 &= -\frac{1}{2}t_0(1-t_1), \\ a_3 &= \frac{1}{3!}(t_0(1-t_1)^2 + t_0^2 t_2), \\ a_4 &= -\frac{1}{4!}(t_0(1-t_1)^3 + 4t_0^2 t_2(1-t_1) + t_0^3 t_3). \end{aligned}$$

To make an analytic analysis, one can fix some  $N > 0$  and take a truncation  $t_n = 0$  for  $n \geq N$ . Then the gradient flow will take any initial value to one of the critical points of  $S$  or to infinity. Under suitable conditions, one can embed the renormalization transformations into the gradient flow and show that the repeated renormalization transformations may take us to the critical point of  $S$  in the limit. For example, take  $t_n = 0$  for  $n \geq 3$ , then the gradient flow equation becomes

$$(13) \quad \frac{\partial x(s)}{\partial s} = t_0 - (1-t_1)x(s) + \frac{1}{2}t_2 x(s)^2.$$

It can be solved by

$$(14) \quad x(s) = -\frac{2t_0 \tan(\frac{1}{2}\alpha s)}{\alpha - (1-t_1) \tan(\frac{1}{2}\alpha s)},$$

where

$$(15) \quad \alpha = (-(1-t_1)^2 + 2t_0 t_2)^{1/2}.$$

One can check that when

$$(16) \quad s_0 = -\frac{2}{\alpha} \arctan\left(\frac{\alpha}{1-t_1}\right),$$

one has

$$(17) \quad x(s_0) = \frac{t_0}{1-t_1}.$$

**2.7. Limit of the repeated renormalization transformation.** By repeating the Newton algorithm, one gets a sequence  $\{x_n = x_0 + \frac{t_0}{1-t_1} + \dots + \frac{t_0^{(n-1)}}{1-t_1^{(n-1)}}\}$ , and  $\{R^n(t_{-1}, t_0, t_1, \dots) = (t_{-1}^{(n)}, t_0^{(n)}, t_1^{(n)}, \dots)\}$ . Then  $\{x_n\}$  converges in the adic topology of formal power series to  $x_\infty$  which is the zero of

$$\frac{\partial S}{\partial x} = 0,$$

i.e.,  $x_{(\infty)}$  is a critical point of  $S$ , and  $\{\mathbf{t}^{(n)} = (t_{-1}^{(n)}, t_0^{(n)}, t_1^{(n)}, \dots)\}$  converges to  $\mathbf{t}^{(\infty)}$  which is the Taylor coefficients of  $S$  at  $x = x_\infty$ .

**Proposition 2.1.** *The limit point  $x_\infty$  satisfies the following equation:*

$$(18) \quad x_\infty = \sum_{n \geq 0} t_n \frac{x_\infty^n}{n!}.$$

**Proposition 2.2.** *The following formula for  $x_\infty$  holds:*

$$(19) \quad x_\infty = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{p_1 + \dots + p_k = k-1} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_k}}{p_k!}.$$

*Proof.* This can be proved by Lagrange inversion formula as follows. Consider

$$(20) \quad z = \frac{w}{t_0 + \sum_{n \geq 1} t_n \frac{w^n}{n!}}.$$

This is a series in  $w$  with leading term  $\frac{w}{t_0}$ . Take the inverse series

$$(21) \quad w = \sum_{k \geq 1} a_k z^k$$

by Lagrange inversion formula:

$$\begin{aligned} a_k &= \operatorname{res}_{z=0} \frac{w}{z^{k+1}} dz = -\frac{1}{k} \operatorname{res}_{z=0} w d \frac{1}{z^k} \\ &= \frac{1}{k} \operatorname{res}_{w=0} \frac{1}{z^k} dw = \frac{1}{k} \operatorname{res}_{w=0} \frac{(\sum_{n \geq 0} t_n \frac{w^n}{n!})^k}{w^k} dw \\ &= \frac{1}{k} \sum_{p_1 + \dots + p_k = k-1} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_k}}{p_k!}. \end{aligned}$$

The proof is completed by setting  $z = 1$ . □

**Theorem 2.3.** *The limit  $\mathbf{t}^{(\infty)}$  is given by:*

$$(22) \quad \mathbf{t}^{(\infty)} = \left( \sum_{n=0}^{\infty} \frac{t_n - \delta_{n,1}}{(n+1)!} I_0^{n+1}, 0, I_1 - 1, I_2, I_3, \dots \right).$$

where for  $k \geq 0$ ,

$$(23) \quad I_k = \sum_{n \geq 0} t_{n+k} \frac{x_\infty^n}{n!}.$$

*Proof.* Note  $\mathbf{t}^{(\infty)}$  are just the Taylor coefficients of  $S$  at  $x = x_\infty$  up to constant factors:

$$(24) \quad S = S(x_\infty) + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^n S}{\partial x^n}(x_\infty) \cdot (x - x_\infty).$$

By (2), we have for  $k \geq 1$ ,

$$\frac{\partial^k S}{\partial x^k}(x_\infty) = \sum_{n \geq 0} t_{n+k-1} \frac{x_\infty^n}{n!} - \delta_{k,1} x_\infty - \delta_{k,2} = I_{k-1} - \delta_{k,1} I_0 - \delta_{k,2}.$$

By (18),

$$(25) \quad x_\infty = I_0,$$

and so we have

$$S(x_\infty) = \sum_{n=0}^{\infty} \frac{t_n - \delta_{n,1}}{(n+1)!} x_\infty^{n+1} = \sum_{n=0}^{\infty} \frac{t_n - \delta_{n,1}}{(n+1)!} I_0^{n+1}.$$

□

Recall that our convention is that  $t_{-1} = 1$ , therefore,

$$(26) \quad I_{-1} = \sum_{n=0}^{\infty} t_n \frac{I_0^{n+1}}{(n+1)!}.$$

Therefore,

$$(27) \quad \sum_{n=0}^{\infty} \frac{t_n - \delta_{n,1}}{(n+1)!} I_0^{n+1} = I_{-1} - \frac{I_0^2}{2}.$$

From an algebraic point of view, the formula for  $\mathbf{t}^{(\infty)}$  in the above Theorem is not perfect since its first term contains explicitly the indeterminate  $t_n$ . We will fix this in the next subsection.

**2.8. A change of coordinates on the space of coupling constants.** By (23) and (19), one can express  $I_n$ 's in terms of  $t_m$ 's by a triangular relation, therefore, one can also express  $t_m$ 's in terms of  $I_n$ 's. The formula turns out to be very simple as in the next Proposition.

**Proposition 2.4.** *One can express  $\{t_k\}_{k=0}^{\infty}$  in terms of  $\{I_k\}_{k=0}^{\infty}$  by the following formula:*

$$(28) \quad t_k = \sum_{n=0}^{\infty} \frac{(-1)^n I_0^n}{n!} I_{n+k}.$$

*Proof.* We first rewrite (23) as follows:

$$(29) \quad t_k = I_k - \sum_{n \geq 1} t_{n+k} \frac{I_0^n}{n!}.$$

Then repeatedly apply this as follows:

$$\begin{aligned}
t_0 &= I_0 - \sum_{n \geq 1} t_n \frac{I_0^n}{n!} = I_0 - t_1 I_0 - t_2 \frac{I_0^2}{2!} - t_3 \frac{I_0^3}{3!} - \dots \\
&= I_0 - (I_1 - \sum_{n \geq 1} t_{n+1} \frac{I_0^n}{n!}) I_0 - t_2 \frac{I_0^2}{2!} - t_3 \frac{I_0^3}{3!} - \dots \\
&= I_0 - I_1 I_0 + \sum_{n \geq 2} t_n \left( \frac{1}{(n-1)!} - \frac{1}{n!} \right) I_0^n \\
&= I_0 - I_1 I_0 + \left( \frac{1}{(2-1)!} - \frac{1}{2!} \right) I_2 I_0^2 \\
&\quad + \sum_{n \geq 3} t_n \left( -\frac{1}{(n-2)!} \left( \frac{1}{(2-1)!} - \frac{1}{2!} \right) + \left( \frac{1}{(n-1)!} - \frac{1}{n!} \right) \right) I_0^n \\
&= I_0 - I_1 I_0 + \frac{1}{2!} I_2 I_0^2 \\
&\quad + \sum_{n \geq 3} t_n \left( -\frac{1}{(n-2)!} \cdot \frac{1}{2!} + \frac{1}{(n-1)!} - \frac{1}{n!} \right) I_0^n.
\end{aligned}$$

Repeating this once more,

$$\begin{aligned}
t_0 &= I_0 - I_1 I_0 + \frac{1}{2!} I_2 I_0^2 - \frac{1}{3!} I_3 I_0^3 \\
&\quad + \sum_{n \geq 4} t_n \left( \frac{1}{(n-3)!} \cdot \frac{1}{3!} - \frac{1}{(n-2)!} \cdot \frac{1}{2!} + \frac{1}{(n-1)!} - \frac{1}{n!} \right) I_0^n.
\end{aligned}$$

Now it is clear how the proof for the case of  $t_0$  can be completed by mathematical induction. The proof for the general case of  $t_k$  is exactly the same.  $\square$

As a corollary to Proposition 2.4, we have:

**Theorem 2.5.** *The limit  $\mathbf{t}^{(\infty)}$  is given by:*

$$(30) \quad \mathbf{t}^{(\infty)} = \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1}, 0, I_1 - 1, I_2, I_3, \dots \right).$$

*Proof.* Just combine (22) with (28) as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{t_n - \delta_{n,1}}{(n+1)!} I_0^{n+1} &= \sum_{n=0}^{\infty} \frac{I_0^{n+1}}{(n+1)!} \sum_{m=0}^{\infty} \frac{(-1)^m I_0^m}{m!} I_{m+n} - \frac{I_0^2}{2} \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{(-1)^m}{m!(k-m+1)!} I_k I_0^{k+1} - \frac{I_0^2}{2} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} I_k I_0^{k+1} - \frac{I_0^2}{2} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1}.
\end{aligned}$$

□

**2.9. Jacobian matrices.** As another straightforward corollary to Proposition 2.4, we have:

**Corollary 2.6.** *The Jacobian matrix of the coordinate change from  $\{T_k\}$  to  $\{t_k\}$  is given by:*

$$(31) \quad \frac{\partial t_k}{\partial I_0} = \delta_{k,0} - t_{k+1},$$

$$(32) \quad \frac{\partial t_k}{\partial I_l} = \frac{(-1)^{l-k} I_0^{l-k}}{(l-k)!} H(l-k), \quad l \geq 1,$$

where  $H(x)$  is the Heaviside function:

$$(33) \quad H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

**Corollary 2.7.** *The vector fields  $\{\frac{\partial}{\partial I_l}\}$  can be expressed in terms of the vector fields  $\{\frac{\partial}{\partial t_k}\}$  as follows:*

$$(34) \quad \frac{\partial}{\partial I_0} = \frac{\partial}{\partial t_0} - \sum_{k \geq 0} t_{k+1} \frac{\partial}{\partial t_k},$$

$$(35) \quad \frac{\partial}{\partial I_l} = \sum_{k=0}^l \frac{(-1)^{l-k} I_0^{l-k}}{(l-k)!} \frac{\partial}{\partial t_k}.$$

Now for  $k \geq 0$ ,

$$\frac{\partial I_0}{\partial t_k} = \frac{I_0^k}{k!} + \sum_{n \geq 1} \frac{t_n}{(n-1)!} I_0^{n-1} \cdot \partial_{t_k} I_0 = \frac{I_0^k}{k!} + I_1 \cdot \partial_{t_k} I_0,$$

and so

$$(36) \quad \frac{\partial I_0}{\partial t_k} = \frac{1}{1 - I_1} \frac{I_0^k}{k!}.$$

And recall for  $l \geq 1$ ,

$$(37) \quad I_l = \sum_{n \geq 0} \frac{t_{n+l}}{n!} I_0^n.$$

Therefore,

$$(38) \quad \frac{\partial I_l}{\partial t_k} = \frac{I_{l+1}}{1 - I_1} \frac{I_0^k}{k!} + \frac{I_0^{k-l}}{(k-l)!} H(k-l),$$

where  $H(x)$  is the Heaviside step function. In fact,

$$\begin{aligned} \frac{\partial I_l}{\partial t_k} &= \sum_{n \geq 1} \frac{t_{n+l}}{(n-1)!} I_0^{n-1} \cdot \partial_{t_k} I_0 + \sum_{n \geq 0} \frac{\delta_{n,k-l}}{n!} I_0^n \\ &= \frac{I_{l+1}}{1 - I_1} \frac{I_0^k}{k!} + \frac{I_0^{k-l}}{(k-l)!} H(k-l), \end{aligned}$$

**Proposition 2.8.** *The vector fields  $\{\frac{\partial}{\partial t_k}\}$  can be expressed in terms of the vector fields  $\{\frac{\partial}{\partial I_l}\}$  as follows:*

$$(39) \quad \frac{\partial}{\partial t_k} = \frac{1}{1 - I_1} \frac{I_0^k}{k!} \frac{\partial}{\partial I_0} + \frac{I_0^k}{k!} \sum_{l \geq 1} \frac{I_{l+1}}{1 - I_1} \frac{\partial}{\partial I_l} + \sum_{1 \leq l \leq k} \frac{I_0^{k-l}}{(k-l)!} \frac{\partial}{\partial I_l}.$$

*Proof.*

$$\begin{aligned} \frac{\partial}{\partial t_k} &= \frac{\partial I_0}{\partial t_k} \frac{\partial}{\partial I_0} + \sum_{l \geq 1} \frac{\partial I_l}{\partial t_k} \frac{\partial}{\partial I_l} \\ &= \frac{1}{1 - I_1} \frac{I_0^k}{k!} \frac{\partial}{\partial I_0} + \sum_{l \geq 1} \left( \frac{I_{l+1}}{1 - I_1} \frac{I_0^k}{k!} + \frac{I_0^{k-l}}{(k-l)!} H(k-l) \right) \frac{\partial}{\partial I_l} \\ &= \frac{1}{1 - I_1} \frac{I_0^k}{k!} \frac{\partial}{\partial I_0} + \frac{I_0^k}{k!} \sum_{l \geq 1} \frac{I_{l+1}}{1 - I_1} \frac{\partial}{\partial I_l} + \sum_{1 \leq l \leq k} \frac{I_0^{k-l}}{(k-l)!} \frac{\partial}{\partial I_l}. \end{aligned}$$

□

For example,

$$(40) \quad \frac{\partial}{\partial t_0} = \frac{1}{1 - I_1} \frac{\partial}{\partial I_0} + \sum_{l \geq 1} \frac{I_{l+1}}{1 - I_1} \frac{\partial}{\partial I_l},$$

$$(41) \quad \frac{\partial}{\partial t_1} = \frac{I_0}{1 - I_1} \frac{\partial}{\partial I_0} + \left( \frac{I_2 I_0}{1 - I_1} + 1 \right) \frac{\partial}{\partial I_1} + \sum_{l \geq 2} \frac{I_{l+1} I_0}{1 - I_1} \frac{\partial}{\partial I_l}.$$

**2.10. Determinants and Cramer's rule for the Jacobian matrices.** It is very interesting to relate the two calculations in the preceding subsection. The Jacobian matrices there turn out provide nice examples of matrices of infinite sizes for which one can formally define their determinants and compute their inverse matrices by Cramer's rule.

First of all, the matrix  $(\frac{\partial t_k}{\partial I_l})_{k,l \geq 0}$  has the form:

$$(42) \quad \begin{pmatrix} 1 - t_1 & -t_2 & -t_3 & -t_4 & -t_5 & -t_6 & \cdots \\ -I_0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{I_0^2}{2!} & -I_0 & 1 & 0 & 0 & 0 & \cdots \\ -\frac{I_0^3}{3!} & \frac{I_0^2}{2!} & -I_0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

To define/calculate its determinant, we expand along the first row to get:

$$(43) \quad \det \left( \frac{\partial t_k}{\partial I_l} \right)_{k,l \geq 0} = (1 - t_1) + \sum_{j=2}^{\infty} (-1)^j t_j A_j,$$

where  $A_j$  is the following determinant:

$$A_j = \begin{vmatrix} -I_0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \frac{I_0^2}{2!} & -I_0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ -\frac{I_0^3}{3!} & \frac{I_0^2}{2!} & -I_0 & 1 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \\ \frac{(-1)^{j-1} I_0^{j-1}}{(j-1)!} & \frac{(-1)^{j-2} I_0^{j-2}}{(j-2)!} & \cdots & \cdots & \cdots & -I_0 & 0 & 0 & \cdots \\ \frac{(-1)^j I_0^j}{j!} & \frac{(-1)^{j-1} I_0^{j-1}}{(j-1)!} & \cdots & \cdots & \cdots & \frac{I_0^2}{2} & 1 & 0 & \cdots \\ \frac{(-1)^{j+1} I_0^{j+1}}{j!} & \frac{(-1)^j I_0^j}{j!} & \cdots & \cdots & \cdots & -\frac{I_0^3}{3!} & -I_0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

Naturally we define:

$$(44) \quad A_j = \begin{vmatrix} -I_0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ \frac{I_0^2}{2!} & -I_0 & 1 & 0 & 0 & \cdots & 0 \\ -\frac{I_0^3}{3!} & \frac{I_0^2}{2!} & -I_0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \frac{(-1)^{j-1} I_0^{j-1}}{(j-1)!} & \frac{(-1)^{j-2} I_0^{j-2}}{(j-2)!} & \cdots & \cdots & \cdots & \cdots & -I_0 \end{vmatrix}$$

They can be computed as follows. By expansion along the first column one gets a recursion relation for  $A_j$ :

$$(45) \quad A_j = -(I_0 A_{j-1} + \frac{I_0^2}{2!} A_{j-2} + \cdots + \frac{I_0^{j-1}}{(j-1)!} A_1),$$

where  $A_1 = 1$ . It is clear that

$$(46) \quad A_j = (-1)^{j-1} \frac{I_0^{j-1}}{(j-1)!}.$$

Therefore,

$$(47) \quad \det \left( \frac{\partial t_k}{\partial I_l} \right)_{k,l \geq 0} = (1 - t_1) - \sum_{j=2}^{\infty} t_j \frac{I_0^{j-1}}{(j-1)!} = 1 - I_1.$$

Similarly, one can define and compute the  $(i, j)$ -minors of the Jacobian matrix  $\left( \frac{\partial t_k}{\partial I_l} \right)_{k,l \geq 0}$ .

More interestingly, even though the matrix  $\left( \frac{\partial I_l}{\partial t_k} \right)_{k,l \geq 0}$  does not have a nice simple shape, nevertheless, after some simple row operations (multiplying the first row by  $\frac{I_0^k}{k!}$  and subtracting it from the row indexed by  $k$ ), it takes the following form:

$$(48) \quad \begin{pmatrix} \frac{1}{1-I_1} & \frac{I_2}{1-I_1} & \frac{I_3}{1-I_1} & \frac{I_4}{1-I_1} & \frac{I_5}{1-I_1} & \frac{I_6}{1-I_1} & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & I_0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & \frac{I_0^2}{2!} & I_0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Therefore, one can still formally define and compute the determinant and the inverse matrix of  $\left( \frac{\partial I_l}{\partial t_k} \right)_{k,l \geq 0}$  as above.

**2.11. The manifold of coupling constants and singular behavior of coordinate transformation.** We introduce a manifold of coupling constants, with two coordinate patches, one with  $\{t_k\}_{k \geq 0}$  as local coordinates, the other with  $\{I_l\}_{l \geq 0}$  as local coordinates. The calculation of the determinant of the Jacobian matrix indicates singular behavior of  $I_l$ 's as functions of  $t_k$ 's along the hypersurface defined by  $I_1 = 1$ . We have already shown that

$$S = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1},$$



and  $x = I_0 = x_\infty$  is a critical point of  $S$ :

$$\left. \frac{\partial S}{\partial x} \right|_{x=I_0} = 0.$$

When  $I_1 = 1$ , one also has

$$(49) \quad \left. \frac{\partial^2 S}{\partial x^2} \right|_{x=I_0} = 0.$$

For each  $k > 1$ , a multicritical point of  $S$  order  $k$  is a point  $x = x_c$  where one has

$$(50) \quad \left. \frac{\partial^j S}{\partial x^j} \right|_{x=x_c} = 0, \quad j = 1, \dots, k-1,$$

$$(51) \quad \left. \frac{\partial^k S}{\partial x^k} \right|_{x=x_c} \neq 0.$$

Clearly,  $S$  has a multicritical point of order  $k > 1$  when

$$(52) \quad t_n = \delta_{n,1} + \delta_{n,k}.$$

Write

$$(53) \quad s_n^{(k)} = t_n - \delta_{n,1} - \delta_{n,k}.$$

We rewrite  $S$  as follows:

$$(54) \quad S = -\frac{1}{k!} x^k + \sum_{n \geq 1} s_{n-1}^{(k)} \frac{x^n}{n!}.$$

Then the equation for the critical point (18) can be rewritten as:

$$(55) \quad \frac{x_\infty^k}{k!} = \sum_{n \geq 0} s_n^{(k)} \frac{x_\infty^n}{n!},$$

and the Taylor expansion at  $x = x_\infty$  still has the form:

$$(56) \quad S = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1} + \sum_{n=2}^{\infty} \frac{I_{n-1} - \delta_{n,2}}{n!} (x - I_0)^n.$$

**2.12. Another coordinate system on the space of couple constants.** Let us write down the first few terms of (19):

$$I_0 = t_0 + t_0 t_1 + \left( \frac{t_0^2}{2} t_2 + t_0 t_1^2 \right) + \left( \frac{1}{6} t_0^3 t_3 + \frac{3}{2} t_0^2 t_1 t_2 + t_0 t_1^3 \right) + \dots$$

One can see that

$$(57) \quad \frac{\partial^k I_0}{\partial t_0^k} - \delta_{k,1} = t_k + \dots,$$

and so  $\{\frac{\partial^k I_0}{\partial t_0^k} - \delta_{k,1}\}_{k \geq 0}$  can be also used as a coordinate system on the space of coupling constants. We now find explicit formula for them as formal power series in  $t$ -coordinates. Note

$$(58) \quad I_0 = t_0 + \sum_{k=2}^{\infty} \frac{1}{k} \sum_{m=1}^{k-1} \binom{k}{m} t_0^m \sum_{\substack{p_1+\dots+p_{k-m}=k-1 \\ p_1, \dots, p_{k-m} > 0}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_{k-m}}}{p_{k-m}!}.$$

Take  $\frac{\partial}{\partial t_0}$  on both sides:

$$\begin{aligned} \frac{\partial I_0}{\partial t_0} &= 1 + \sum_{k=2}^{\infty} \frac{1}{k} \sum_{m=1}^{k-1} \binom{k}{m} m t_0^{m-1} \sum_{\substack{p_1+\dots+p_{k-m}=k-1 \\ p_1, \dots, p_{k-m} > 0}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_{k-m}}}{p_{k-m}!} \\ &= 1 + \sum_{k=2}^{\infty} \sum_{m=1}^{k-1} \binom{k-1}{m-1} t_0^{m-1} \sum_{\substack{p_1+\dots+p_{k-m}=k-1 \\ p_1, \dots, p_{k-m} > 0}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_{k-m}}}{p_{k-m}!} \\ &= 1 + t_1 + \sum_{k=1}^{\infty} \sum_{m=0}^{k-1} \binom{k}{m} t_0^m \sum_{\substack{p_1+\dots+p_{k-m}=k \\ p_1, \dots, p_{k-m} > 0}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_{k-m}}}{p_{k-m}!} \\ &= 1 + \sum_{k=1}^{\infty} \sum_{p_1+\dots+p_k=k} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_k}}{p_k!}. \end{aligned}$$

Once more,

$$\begin{aligned} \frac{\partial^2 I_0}{\partial t_0^2} &= \sum_{k=2}^{\infty} \sum_{m=0}^{k-1} \binom{k}{m} m t_0^{m-1} \sum_{\substack{p_1+\dots+p_{k-m}=k \\ p_1, \dots, p_{k-m} > 0}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_{k-m}}}{p_{k-m}!} \\ &= t_2 + \sum_{k=3}^{\infty} k \sum_{m=1}^{k-1} \binom{k-1}{m-1} t_0^{m-1} \sum_{\substack{p_1+\dots+p_{k-m}=k \\ p_1, \dots, p_{k-m} > 0}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_{k-m}}}{p_{k-m}!} \\ &= t_2 + \sum_{k=2}^{\infty} (k+1) \sum_{m=0}^{k-1} \binom{k}{m} t_0^m \sum_{\substack{p_1+\dots+p_{k-m}=k+1 \\ p_1, \dots, p_{k-m} > 0}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_{k-m}}}{p_{k-m}!} \\ &= t_2 + \sum_{k=2}^{\infty} (k+1) \sum_{p_1+\dots+p_k=k+1} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_k}}{p_k!}. \end{aligned}$$

and inductively, one finds

$$(59) \quad \frac{\partial^l I_0}{\partial t_0^l} = \sum_{k=1}^{\infty} (k+1) \cdots (k+l-1) \sum_{p_1+\cdots+p_k=k+l-1} \frac{t_{p_1}}{p_1!} \cdots \frac{t_{p_k}}{p_k!}.$$

**2.13. Derivatives of  $I_0$  and the  $I$ -coordinates.** In this subsection we present explicit formulas for the coordinate changes between  $\{\frac{\partial^k I_0}{\partial t_0^k} - \delta_{k,1}\}_{k \geq 0}$  and  $\{I_n\}_{n \geq 0}$ . Recall by (36) and (38), we have

$$(60) \quad \frac{\partial I_0}{\partial t_0} = \frac{1}{1 - I_1},$$

and for  $l \geq 1$ ,

$$(61) \quad \frac{\partial I_l}{\partial t_0} = \frac{I_{l+1}}{1 - I_1}.$$

Taking derivatives on both sides of (60) and using (61) repeatedly, one gets:

$$\begin{aligned} \frac{\partial^2 I_0}{\partial t_0^2} &= \frac{I_2}{(1 - I_1)^3}, \\ \frac{\partial^3 I_0}{\partial t_0^3} &= \frac{I_3}{(1 - I_1)^4} + \frac{3I_2^2}{(1 - I_1)^5}, \\ \frac{\partial^4 I_0}{\partial t_0^4} &= \frac{I_4}{(1 - I_1)^5} + \frac{10I_2I_3}{(1 - I_1)^6} + \frac{15I_2^3}{(1 - I_1)^7}. \end{aligned}$$

In general we have:

**Proposition 2.9.** *The higher derivatives in  $t_0$  can be written in  $I$ -coordinates as follows:*

$$(62) \quad \frac{\partial^n I_0}{\partial t_0^n} = \sum_{\sum_{j \geq 1} j m_j = n-1} \frac{(\sum_j (j+1)m_j)!}{\prod_j ((j+1)!)^{m_j} m_j!} \cdot \frac{\prod_j I_{j+1}^{m_j}}{(1 - I_1)^{\sum_j (j+1)m_j + 1}}.$$

*Proof.* Proof by induction using (40).  $\square$

Conversely, one can also express the  $I$ -coordinates in terms  $\frac{\partial^k I_0}{\partial t_0^k}$ . By (60), we have

$$I_1 = 1 - \frac{1}{\frac{\partial I_0}{\partial t_0}}.$$

Take  $\frac{\partial}{\partial t_0}$  on both sides and use (61) for  $l = 1$ :

$$I_2 = (1 - I_1) \cdot \frac{\frac{\partial^2 I_0}{\partial t_0^2}}{\left(\frac{\partial I_0}{\partial t_0}\right)^2} = \frac{\frac{\partial^2 I_0}{\partial t_0^2}}{\left(\frac{\partial I_0}{\partial t_0}\right)^3}.$$

Repeating this process for several times, one gets

$$I_3 = \frac{\frac{\partial^3 I_0}{\partial t_0^3}}{\left(\frac{\partial I_0}{\partial t_0}\right)^4} - 3 \frac{\left(\frac{\partial^2 I_0}{\partial t_0^2}\right)^2}{\left(\frac{\partial I_0}{\partial t_0}\right)^5},$$

$$I_4 = \frac{\frac{\partial^4 I_0}{\partial t_0^4}}{\left(\frac{\partial I_0}{\partial t_0}\right)^5} - 10 \frac{\frac{\partial^2 I_0}{\partial t_0^2} \frac{\partial^3 I_0}{\partial t_0^3}}{\left(\frac{\partial I_0}{\partial t_0}\right)^6} + 15 \frac{\left(\frac{\partial^2 I_0}{\partial t_0^2}\right)^3}{\left(\frac{\partial I_0}{\partial t_0}\right)^7}.$$

In general we have:

**Proposition 2.10.** *The  $I$ -coordinates can be written in the higher derivatives in  $t_0$  as follows:*

$$(63) \quad I_n = - \sum_{\sum_{j \geq 1} j m_j = n-1} \frac{(\sum_j (j+1) m_j)!}{\prod_j ((j+1)!)^{m_j} m_j!} \cdot \frac{\prod_j \left(-\frac{\partial^{j+1} I_0}{\partial t_0^{j+1}}\right)^{m_j}}{\left(\frac{\partial I_0}{\partial t_0}\right)^{\sum_j (j+1) m_j + 1}}.$$

*Proof.* Essentially the same as the proof of Proposition 2.9.  $\square$

Now we combine (64) with (63) to get:

$$(64) \quad t_k = - \sum_{n=0}^{\infty} \frac{(-1)^n I_0^n}{n!} \sum_{\sum_{j \geq 1} j m_j = n+k-1} \frac{(\sum_j (j+1) m_j)!}{\prod_j ((j+1)!)^{m_j} m_j!}$$

$$\cdot \frac{\prod_j \left(\frac{\partial^{j+1} I_0}{\partial t_0^{j+1}}\right)^{m_j}}{\left(-\frac{\partial I_0}{\partial t_0}\right)^{\sum_j (j+1) m_j + 1}}.$$

This formula expresses the  $t$ -coordinates in terms of derivatives of  $I_0$  in  $t_0$ .

### 3. FEYNMAN RULES FOR $I_k$

In this Section, we will first propose Feynman rules for  $I_0$ , and based on them, propose Feynman rules for  $I_k$  ( $k > -1$ ) and  $I_{-1} - \frac{1}{2} I_0^2$ . We will prove these rules in a later Section.

**3.1. Feynman rules for  $I_0$ .** The following are the first few terms of  $I_0 = x_\infty$ :

$$\begin{aligned}
x_\infty = & t_0 + t_1 t_0 + \left( t_2 \frac{t_0^2}{2!} + 2 \frac{t_1^2}{2!} t_0 \right) + \left( t_3 \frac{t_0^3}{3!} + 3 t_1 t_2 \frac{t_0^2}{2!} + 6 \frac{t_1^3}{3!} t_0 \right) \\
& + \left[ t_4 \frac{t_0^4}{4!} + \left( 6 \frac{t_2^2}{2!} + 4 t_1 t_3 \right) \frac{t_0^3}{3!} + 12 t_2 \frac{t_1^2}{2!} \frac{t_0^2}{2!} + 24 \frac{t_1^4}{4!} t_0 \right] \\
& + \left[ t_5 \frac{t_0^5}{5!} + (5 t_1 t_4 + 10 t_2 t_3) \frac{t_0^4}{4!} + \left( 30 t_1 \frac{t_2^2}{2!} + 20 t_3 \frac{t_1^2}{2!} \right) \frac{t_0^3}{3!} \right. \\
& \quad \left. + 60 t_2 \frac{t_1^3}{3!} \frac{t_0^2}{2!} + 120 \frac{t_1^5}{5!} t_0 \right] \\
& + \left[ t_6 \frac{t_0^6}{6!} + \left( 20 \frac{t_3^2}{2!} + 6 t_1 t_5 + 15 t_2 t_4 \right) \frac{t_0^5}{5!} \right. \\
& \quad + \left( 90 \frac{t_2^3}{3!} + 30 t_4 \frac{t_1^2}{2!} + 60 t_1 t_2 t_3 \right) \frac{t_0^4}{4!} \\
& \quad \left. + \left( 120 t_3 \frac{t_1^3}{3!} + 180 \frac{t_1^2 t_2^2}{2! 2!} \right) \frac{t_0^3}{3!} + 360 t_2 \frac{t_1^4}{4!} \frac{t_0^2}{2!} + 720 \frac{t_1^6}{6!} t_0 \right] + \dots
\end{aligned}$$

By looking at these explicit expressions, one can formulate the following

**Theorem 3.1.** *The formal power series  $I_0$  is given by a sum over rooted trees*

$$(65) \quad x_\infty = \sum_{\Gamma \text{ is a rooted tree}} \frac{1}{|\text{Aut } \Gamma|} w_\Gamma,$$

where the weight of  $\Gamma$  is given by

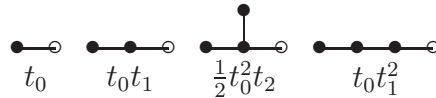
$$(66) \quad w_\Gamma = \prod_{v \in V(\Gamma)} w_v \cdot \prod_{e \in E(\Gamma)} w_e,$$

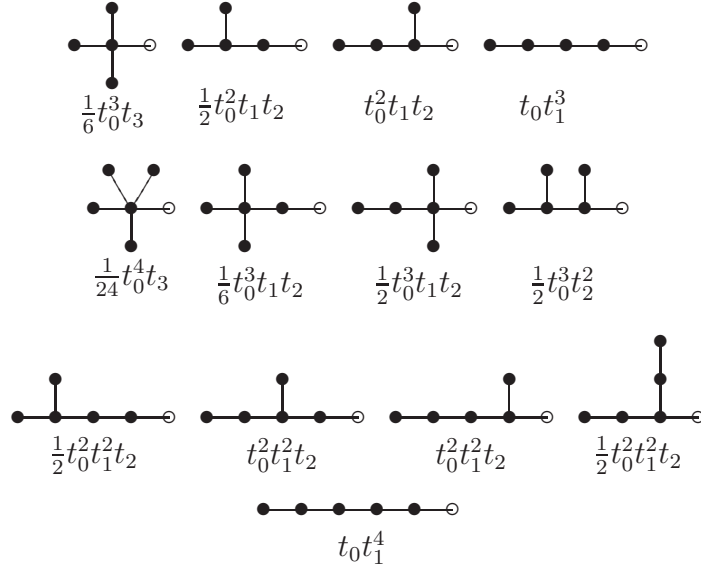
with  $w_e$  and  $w_v$  given by the following Feynman rule:

$$(67) \quad w(e) = 1,$$

$$(68) \quad w(v) = \begin{cases} t_{\text{val}(v)-1}, & \text{if } v \text{ is not the root vertex } \circ, \\ 1, & \text{if } v \text{ is the root vertex } \circ. \end{cases}$$

For example,



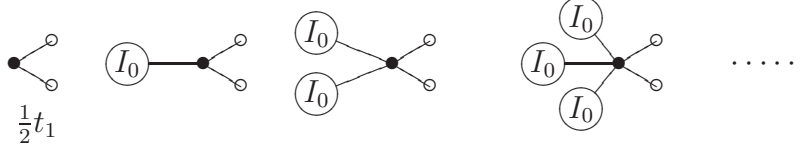


The sum of contributions of all such diagrams can be symbolically denoted by  $(I_0) \longrightarrow \circ$ . We will prove this Theorem later in Section ??.

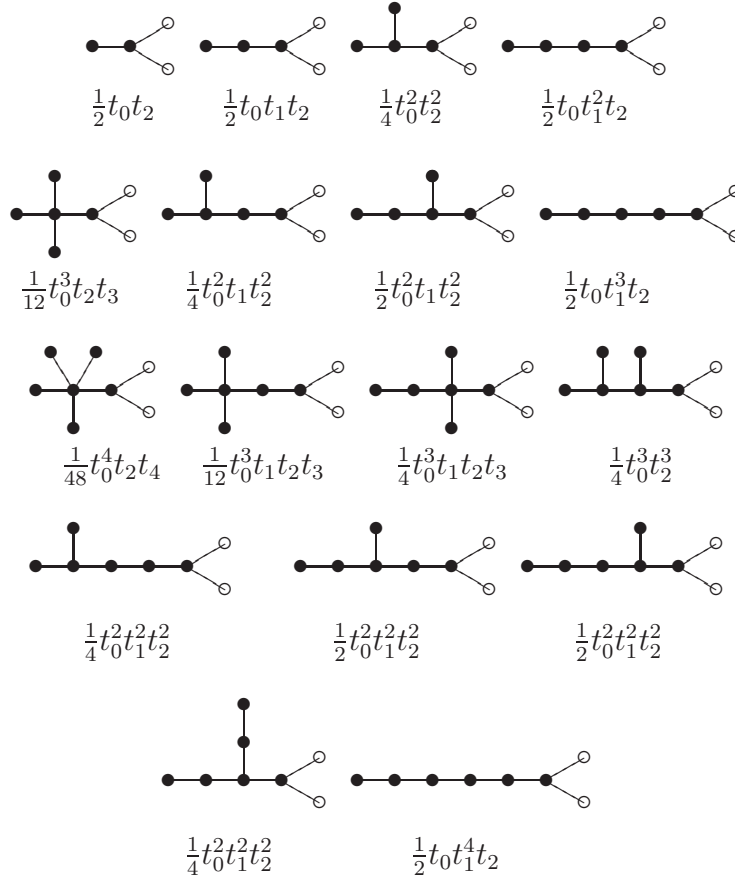
**3.2. Feynman rules for  $I_k$ .** By (23) and the above Feynman rules for  $I_0$ , one can express each  $I_k$  as a sum over some Feynman diagrams. For example,

$$\begin{aligned}
\frac{I_1}{2} &= \frac{1}{2}t_1 + \frac{1}{2} \sum_{n \geq 1} t_{n+1} \frac{I_0^n}{n!} \\
&= \frac{1}{2}t_1 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{t_{n+1}}{n!} \left( t_0 + t_0t_1 + \left( \frac{1}{2}t_0^2t_2 + t_0t_1^2 \right) + \left( \frac{1}{6}t_0^3t_3 + \frac{3}{2}t_0^2t_1t_2 + t_0t_1^3 \right) \right. \\
&\quad \left. + \left( \frac{1}{24}t_0^4t_4 + \frac{2}{3}t_0^3t_1t_3 + \frac{1}{2}t_0^3t_2^2 + 3t_0^2t_1^2t_2 + t_0t_1^4 \right) + \cdots \right)^n \\
&= \frac{1}{2}t_1 + \frac{1}{2}t_0t_2 + \left( \frac{1}{2}t_0t_1t_2 + \frac{1}{4}t_0^2t_3 \right) \\
&\quad + \left( \frac{1}{4}t_0^2t_2^2 + \frac{1}{2}t_0t_1^2t_2 + \frac{1}{2}t_0^2t_1t_3 + \frac{1}{12}t_0^3t_4 \right) \\
&\quad + \left( \frac{1}{3}t_0^3t_2t_3 + \frac{3}{4}t_0^2t_1t_2^2 + \frac{1}{2}t_0t_1^3t_2 + \frac{3}{4}t_0^2t_1^2t_3 + \frac{1}{4}t_0^3t_1t_4 + \frac{1}{48}t_0^4t_5 \right) \\
&\quad + \left( \frac{1}{2}t_0t_1^4t_2 + \frac{3}{2}t_0^2t_1^2t_2^2 + \frac{1}{4}t_0^3t_2^3 + \frac{4}{3}t_0^3t_1t_2t_3 + t_0^2t_1^3t_3 \right. \\
&\quad \left. + \frac{7}{48}t_0^4t_2t_4 + \frac{1}{12}t_0^4t_3^2 + \frac{1}{2}t_0^3t_1^2t_4 + \frac{1}{12}t_0^4t_1t_5 + \frac{1}{240}t_0^5t_6 \right) + \cdots
\end{aligned}$$

The Feynman diagrams for  $\frac{I_1}{2!}$  are:



where for example,  $(I_0) \text{---} \bullet \text{---} \circ \text{---} \circ$  stands for the sum of all the Feynman diagrams of the following form:



These diagrams can be obtained by grafting the diagrams in  $(I_0) \text{---} \circ$  to  $\circ \text{---} \bullet \text{---} \circ \text{---} \circ$ . Similarly for other diagrams.

*Definition 3.2.* By a tree of type  $k$  we mean a rooted tree whose vertices are marked either by  $\bullet$  or  $\circ$ , the root vertex  $v_0$  is marked by  $\bullet$ , and there are exactly  $k$  vertices marked by  $\circ$ , all of which are adjacent to  $v_0$ .

**Theorem 3.3.** *The formal power series  $I_k$  is given by a sum over rooted trees of type  $k$ :*

$$(69) \quad I_k = \sum_{\Gamma \text{ is a rooted tree of type } k} \frac{1}{|\text{Aut } \Gamma|} w_\Gamma,$$

where the weight of  $\Gamma$  is given by

$$(70) \quad w_\Gamma = \prod_{v \in V(\Gamma)} w_v \cdot \prod_{e \in E(\Gamma)} w_e,$$

with  $w_e$  and  $w_v$  given by the following Feynman rule:

$$(71) \quad w(e) = 1,$$

$$(72) \quad w(v) = \begin{cases} t_{\text{val}(v)-1}, & \text{if } v \text{ is not the root vertex } \circ, \\ 1, & \text{if } v \text{ is the root vertex } \circ. \end{cases}$$

*Proof.* This is clear from the formula

$$(73) \quad I_k = \sum_{n \geq 0} t_{n+k} \frac{I_0^n}{n!}$$

and Theorem 3.1. □

### 3.3. Feynman rules for $I_{-1}$ .

**Theorem 3.4.** *The formal power series  $I_{-1}$  is given by a sum over rooted trees:*

$$(74) \quad I_{-1} = \sum_{\Gamma \text{ is a rooted tree}} \frac{1}{|\text{Aut } \Gamma|} w_\Gamma,$$

where the weight of  $\Gamma$  is given by

$$(75) \quad w_\Gamma = \prod_{v \in V(\Gamma)} w_v \cdot \prod_{e \in E(\Gamma)} w_e,$$

with  $w_e$  and  $w_v$  given by the following Feynman rule:

$$(76) \quad w(e) = 1,$$

$$(77) \quad w(v) = t_{\text{val}(v)-1},$$

*Proof.* This is clear from the formula

$$(78) \quad I_{-1} = \sum_{n=0}^{\infty} t_n \frac{I_0^{n+1}}{(n+1)!}$$

and Theorem 3.1. □



### 3.4. Feynman rules for $I_{-1} - \frac{1}{2}I_0^2$ .

**Proposition 3.5.** *We have*

$$(79) \quad \frac{\partial}{\partial t_0}(I_{-1} - \frac{1}{2}I_0^2) = I_0.$$

*Proof.* Recall

$$I_{-1} - \frac{1}{2}I_0^2 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1},$$

and we have

$$\frac{\partial}{\partial t_0} = \frac{1}{1-I_1} \frac{\partial}{\partial I_0} + \sum_{l \geq 1} \frac{I_{l+1}}{1-I_1} \frac{\partial}{\partial I_l}.$$

The proof is completed by a simple calculation.  $\square$

As a corollary to this Proposition and Theorem 3.1, we have:

**Theorem 3.6.** *The formal power series  $I_{-1}$  is given by a sum over rooted trees:*

$$(80) \quad I_{-1} - \frac{1}{2}I_0^2 = \sum_{\Gamma \text{ is a tree}} \frac{1}{|\text{Aut } \Gamma|} w_{\Gamma},$$

where the weight of  $\Gamma$  is given by

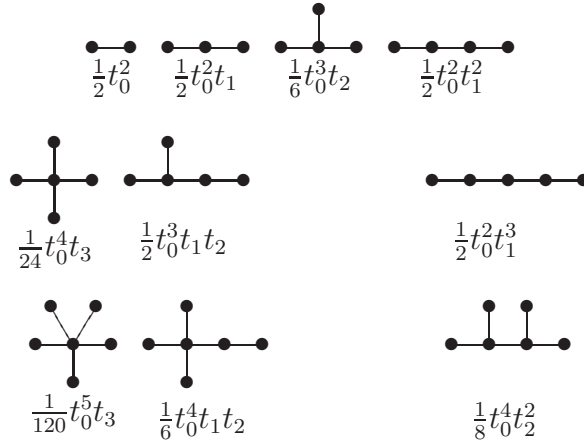
$$(81) \quad w_{\Gamma} = \prod_{v \in V(\Gamma)} w_v \cdot \prod_{e \in E(\Gamma)} w_e,$$

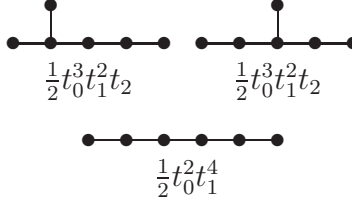
with  $w_e$  and  $w_v$  given by the following Feynman rules:

$$(82) \quad w(e) = 1,$$

$$(83) \quad w(v) = t_{\text{val}(v)-1},$$

For example,





These diagrams give the first few terms of  $I_{-1} - \frac{1}{2}I_0^2$ :

$$\begin{aligned}
I_{-1} - \frac{1}{2}I_0^2 &= \frac{1}{2}t_0^2 + \frac{1}{2}t_0^2 t_1 + \frac{1}{6}t_0^3 t_2 + \frac{1}{2}t_0^2 t_1^2 \\
&+ \frac{1}{24}t_0^4 t_3 + \frac{1}{2}t_0^3 t_1 t_2 + \frac{1}{2}t_0^2 t_1^3 \\
&+ \frac{1}{120}t_0^5 t_3 + \frac{1}{6}t_0^4 t_1 t_2 + \frac{1}{8}t_0^4 t_2^2 \\
&+ \frac{1}{2}t_0^3 t_1^2 t_2 + \frac{1}{2}t_0^3 t_1^2 t_2 + \frac{1}{2}t_0^2 t_1^4 + \cdots
\end{aligned}$$

**3.5. Feynman rules for  $t_k$ .** By (28) we have:

$$\frac{1}{(k+1)!}t_k = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(k+1)!} I_0^n I_{n+k}.$$

The right-hand side is a summation over trees with one  $\bullet$ -vertex of valence  $n+k+1$ , on which  $n$  edges are attached, connecting to  $n$   $\bullet$ -vertex of valence 1, and  $k+1$  edges are attached, connecting to  $k+1$   $\circ$ -vertex of valence 1. One can easily formulate Feynman rules for the right-hand side, a term  $\frac{I_0^n}{n!} I_{n+k}$  corresponds to a rooted tree with  $n$ , the rules for the weights are:

$$(84) \quad w_v = \begin{cases} -I_0, & \text{if } v \text{ is a } \bullet\text{-vertex of valence 1,} \\ I_{n+k+1}, & \text{if } v \text{ is the } \bullet\text{-vertex of valence } n+k+1, \\ 1, & \text{if } v \text{ is a } \circ\text{-vertex of valence 1,} \end{cases}$$

$$(85) \quad w_e = 1.$$

The factor  $n!(k+1)!$  is exactly the order of the automorphism group of this tree.

#### 4. MEAN FIELD THEORY OF THE TOPOLOGICAL 1D GRAVITY

The Feynman diagrams and Feynman rules in last Section suggests that they come from some quantum field theory. In this Section we propose that they arise in the mean field theory of the topological 1D gravity.

**4.1. Gaussian integrals and some properties.** Recall the Gaussian integrals ( $a > 0$ ):

$$(86) \quad \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx x^n e^{-\frac{a}{2}x^2} = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \frac{(2m)!}{m!2^m a^{m+1/2}} = \frac{(2m-1)!!}{a^{m+1/2}}, & \text{if } n = 2m. \end{cases}$$

In this paper, we will use some properties of the Gaussian integrals summarized in the following:

**Proposition 4.1.** *Gaussian integrals have the following properties:*

(1) *(Scaling of variable)* For  $a, b > 0$ ,

$$(87) \quad \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \cdot x^n e^{-\frac{a}{2}x^2} = \left(\frac{b}{a}\right)^{(n+1)/2} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \cdot x^n e^{-\frac{b}{2}x^2}.$$

(2) *(Translation of variable)* For  $a > 0$  and any  $c \in \mathbb{R}$ ,

$$(88) \quad \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \cdot x^n e^{-\frac{a}{2}x^2} = e^{-\frac{ac^2}{2}} \sum_{j=0}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx (x+c)^n (-ac)^j \frac{x^j}{j!} \cdot e^{-\frac{a}{2}x^2}.$$

(3) *(Separation of the square term)* When  $a > 0$  and  $a+b > 0$ ,

$$(89) \quad \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \cdot x^n e^{-\frac{a+b}{2}x^2} = \sum_{j \geq 0} \frac{b^j}{2^j j!} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \cdot x^{n+2j} e^{-\frac{a}{2}x^2}.$$

(4) *(Integration by parts)* For  $a > 0$ ,

$$(90) \quad \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \cdot \partial_x \left( x^n e^{-\frac{a}{2}x^2} \right) = 0.$$

*Proof.* These easily follow from ordinary properties of integrals. However, for our purpose, we will need proofs based on (86) only. It is clear that (87) follows from (86). Now we prove (88). First let  $n = 2m$ , then

$$\begin{aligned} & e^{-\frac{ac^2}{2}} \sum_{j=0}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx (x+c)^n (-ac)^j \frac{x^j}{j!} \cdot e^{-\frac{a}{2}x^2} \\ &= e^{-\frac{ac^2}{2}} \sum_{j=0}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \sum_{k=0}^n \binom{n}{k} x^k c^{n-k} \cdot (-ac)^j \frac{x^j}{j!} \cdot e^{-\frac{a}{2}x^2} \\ &= e^{-\frac{ac^2}{2}} \cdot \sum_{k=0}^n \binom{n}{k} c^{n-k} \cdot \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \cdot \sum_{j=0}^{\infty} (-ac)^j \frac{x^{j+k}}{j!} \cdot e^{-\frac{a}{2}x^2}. \end{aligned}$$

When  $k = 2l$ ,

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \cdot \sum_{j=0}^{\infty} (-ac)^j \frac{x^{j+2l}}{j!} \cdot e^{-\frac{a}{2}x^2} \\
&= \sum_{j=0}^{\infty} \frac{(-ac)^{2j}}{(2j)!} \frac{(2j+2l-1)!!}{a^{j+l+1/2}} \\
&= \frac{1}{a^{l+1/2}} \sum_{j=0}^{\infty} \frac{(ac^2)^j}{2^j j!} \frac{(2j+2l-1)!!}{(2j-1)!!} \\
&= \frac{1}{a^{l+1/2}} \sum_{j=0}^l \frac{(2l)!}{(2l-2j)! j! 2^j} (ac^2)^{l-j} \cdot e^{ac^2/2};
\end{aligned}$$

when  $k = 2l + 1$ ,

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \cdot \sum_{j=0}^{\infty} (-ac)^j \frac{x^{j+2l+1}}{j!} \cdot e^{-\frac{a}{2}x^2} \\
&= \sum_{j=0}^{\infty} \frac{(-ac)^{2j+1}}{(2j+1)!} \frac{(2j+2l+1)!!}{a^{j+l+3/2}} \\
&= -\frac{c}{a^{l+1/2}} \sum_{j=0}^{\infty} \frac{(ac^2)^j}{2^j j!} \frac{(2j+2l+1)!!}{(2j+1)!!} \\
&= -\frac{c}{a^{l+1/2}} \sum_{j=0}^l \frac{(2l+1)!}{(2l+1-2j)! j! 2^j} (ac^2)^{l-j} \cdot e^{ac^2/2}.
\end{aligned}$$

In the above we have used the identities in Lemma 4.2. When  $n = 2m$ ,

$$\begin{aligned}
& e^{-\frac{ac^2}{2}} \sum_{j=0}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx (x+c)^n (-ac)^j \frac{x^j}{j!} \cdot e^{-\frac{a}{2}x^2} \\
&= \sum_{l=0}^m \binom{2m}{2l} c^{2m-2l} \cdot \frac{1}{a^{l+1/2}} \sum_{j=0}^l \frac{(2l)!}{(2l-2j)!j!2^j} (ac^2)^{l-j} \\
&\quad - \sum_{l=0}^{m-1} \binom{2m}{2l+1} c^{2m-2l-1} \cdot \frac{c}{a^{l+1/2}} \sum_{j=0}^l \frac{(2l+1)!}{(2l+1-2j)!j!2^j} (ac^2)^{l-j} \\
&= \sum_{j=0}^m \frac{c^{2m-2j}}{a^{j+1/2}} (2j-1)!! \sum_{l=j}^m \binom{2m}{2l} \cdot \frac{(2l)!}{(2l-2j)!(2j)!} \\
&\quad - \sum_{j=0}^{m-1} \frac{c^{2m-2j}}{a^{j+1/2}} (2j-1)!! \sum_{l=j}^{m-1} \binom{2m}{2l+1} \cdot \frac{(2l+1)!}{(2l+1-2j)!(2j)!} \\
&= \frac{(2m-1)!!}{a^{m+1/2}} + \sum_{j=0}^{m-1} \frac{c^{2m-2j}}{a^{j+1/2}} (2j-1)!! \\
&\quad \cdot \left( \sum_{k=0}^{m-j} \binom{2m}{2m-2j-2k, 2j, 2k} - \sum_{k=0}^{m-j-1} \binom{2m}{2m-2j-2k-1, 2k+1, 2j} \right) \\
&= \frac{(2m-1)!!}{a^{m+1/2}},
\end{aligned}$$

in the last equality we have used the fact that

$$\sum_{k=0}^{m-j} \binom{2m}{2m-2j-2k, 2j, 2k} - \sum_{k=0}^{m-j-1} \binom{2m}{2m-2j-2k-1, 2k+1, 2j} = 0$$

because it is the coefficient of  $z^{2j}$  in the expansion of  $(1 - 1 + z)^{2m}$ . Similarly, when  $n = 2m + 1$ ,

$$\begin{aligned}
& e^{-\frac{ac^2}{2}} \sum_{j=0}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx (x+c)^n (-ac)^j \frac{x^j}{j!} \cdot e^{-\frac{a}{2}x^2} \\
&= \sum_{l=0}^m \binom{2m+1}{2l} c^{2m+1-2l} \cdot \frac{1}{a^{l+1/2}} \sum_{j=0}^l \frac{(2l)!}{(2l-2j)!j!2^j} (ac^2)^{l-j} \\
&- \sum_{l=0}^m \binom{2m+1}{2l+1} c^{2m-2l} \cdot \frac{c}{a^{l+1/2}} \sum_{j=0}^l \frac{(2l+1)!}{(2l+1-2j)!j!2^j} (ac^2)^{l-j} \\
&= \sum_{j=0}^m \frac{c^{2m+1-2j}}{a^{j+1/2}} (2j-1)!! \sum_{l=j}^m \binom{2m+1}{2l} \cdot \frac{(2l)!}{(2l-2j)!(2j)!} \\
&- \sum_{j=0}^m \frac{c^{2m+1-2j}}{a^{j+1/2}} (2j-1)!! \sum_{l=j}^m \binom{2m+1}{2l+1} \cdot \frac{(2l+1)!}{(2l+1-2j)!(2j)!} \\
&= \sum_{j=0}^m \frac{c^{2m-2j}}{a^{j+1/2}} (2j-1)!! \\
&\quad \cdot \left( \sum_{k=0}^{m-j} \binom{2m+1}{2m-2j-2k+1, 2j, 2k} - \sum_{k=0}^{m-j} \binom{2m+1}{2m-2j-2k, 2k+1, 2j} \right) \\
&= 0,
\end{aligned}$$

in the last equality we have used the fact that

$$\sum_{k=0}^{m-j} \binom{2m+1}{2m+1-2j-2k, 2j, 2k} - \sum_{k=0}^{m-j} \binom{2m}{2m-2j-2k, 2k+1, 2j} = 0$$

because it is the coefficient of  $z^{2j}$  in the expansion of  $(1 - 1 + z)^{2m+1}$ .

This completes the proof of (88).

When  $n = 2m$ ,

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \cdot x^{2m} e^{-\frac{a+b}{2}x^2} = \frac{(2m-1)!!}{(a+b)^{m+1/2}}, \\
& \sum_{j \geq 0} \frac{b^j}{2^j j!} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \cdot x^{2m+2j} e^{-\frac{a}{2}x^2} = \sum_{j \geq 0} \frac{b^j}{2^j j!} \frac{(2m+2j-1)!!}{a^{m+j+1/2}},
\end{aligned}$$

so one need to check that

$$\frac{a^{m+1/2}}{(a+b)^{m+1/2}} = \sum_{j \geq 0} \frac{(2m+2j-1)!!}{2^j j! (2m-1)!!} \frac{b^j}{a^j},$$

but this is just a special case of Taylor expansion. When  $n = 2m + 1$ ,

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \cdot x^{2m+1} e^{-\frac{a+b}{2}x^2} = 0, \\ & \sum_{j \geq 0} \frac{b^j}{2^j j!} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \cdot x^{2m+1+2j} e^{-\frac{a}{2}x^2} = 0. \end{aligned}$$

Therefore, we have proved (89).

When  $n = 2m$

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \cdot \partial_x \left( x^{2m} e^{-\frac{a}{2}x^2} \right) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \cdot (2mx^{2m-1} - ax^{2m+1}) e^{-\frac{a}{2}x^2} \frac{1}{\sqrt{2\pi}} = 0; \end{aligned}$$

when  $n = 2m + 1$ ,

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \cdot \partial_x \left( x^{2m+1} e^{-\frac{a}{2}x^2} \right) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \cdot ((2m+1)x^{2m} - ax^{2m+2}) e^{-\frac{a}{2}x^2} \frac{1}{\sqrt{2\pi}} \\ &= (2m+1) \cdot \frac{(2m-1)!!}{a^{m+1/2}} - a \cdot \frac{(2m+1)!!}{a^{m+1+1/2}} = 0. \end{aligned}$$

So (90) is proved.  $\square$

**Lemma 4.2.** For  $l \geq 0$ ,

$$(91) \quad \sum_{j \geq 0} \frac{x^j}{j! 2^j} \frac{(2j+2l-1)!!}{(2j-1)!!} = \sum_{j=0}^l \frac{(2l)!}{(2l-2j)! j! 2^j} x^{l-j} \cdot e^{x/2},$$

$$(92) \quad \sum_{j \geq 0} \frac{x^j}{j! 2^j} \frac{(2j+2l+1)!!}{(2j+1)!!} = \sum_{j=0}^l \frac{(2l+1)!}{(2l+1-2j)! j! 2^j} x^{l-j} \cdot e^{x/2}.$$

*Proof.* Note,

$$\sum_{j \geq 0} \frac{x^j}{j! 2^j} \frac{(2j+2k-1)!!}{(2j-1)!!} = \left( 2x \frac{d}{dx} + 2k-1 \right) \cdots \left( 2x \frac{d}{dx} + 1 \right) e^{x/2}.$$

It follows that

$$\sum_{j \geq 0} \frac{x^j}{j! 2^j} \frac{(2j+2k-1)!!}{(2j-1)!!} = p_k(x) e^{x/2}$$

for some polynomial of degree  $k$ , and  $p_0(x) = 1$ . Furthermore, from

$$p_k(x) e^{x^2/2} = \left( 2x \frac{d}{dx} + 2k-1 \right) (p_{k-1}(x) e^{x^2/2})$$

one derives a recursion relation:

$$p_k(x) = 2xp'_{k-1}(x) + (2k-1)p_{k-1}(x) + xp_{k-1}(x).$$

It is straightforward to check that

$$p_k(x) = \sum_{j=0}^k \frac{(2k)!}{(2k-2j)!j!2^j} x^{k-j}$$

is the unique solution of this recursion relation with initial value. This proves the first identity. The second identity can be proved in the same way.  $\square$

**4.2. Polymer model.** Consider the formal Gaussian integral:

$$(93) \quad Z = \frac{1}{\sqrt{2\pi\lambda}} \int dx \exp \frac{1}{\lambda^2} \left( -\frac{1}{2}x^2 + \sum_{n \geq 1} t_{n-1} \frac{x^n}{n!} \right).$$

This is defined by first expanding  $\exp \frac{1}{\lambda^2} \left( \sum_{n \geq 1} t_{n-1} \frac{x^n}{n!} \right)$  as a formal power series in  $x$  then taking the Gaussian integrals term by term. After a change of variables:

$$(94) \quad Z = \frac{1}{\sqrt{2\pi}} \int dx \exp \left( -\frac{1}{2}x^2 + \sum_{n \geq 1} \frac{t_{n-1}}{n!} \lambda^{n-2} x^n \right).$$

As is well-known, the asymptotic expansion is given by the following summation over Feynman diagrams:

$$(95) \quad Z = \sum_{\Gamma \in \mathcal{G}} \frac{1}{|\text{Aut}(\Gamma)|} \prod_{v \in V(\Gamma)} \lambda^{\text{val}(v)-2} t_{\text{val}(v)-1},$$

where the sum is taken over the set  $\mathcal{G}$  all possible graphs, with the following Feynman rules:

$$(96) \quad w(v) = \lambda^{\text{val}(v)-2} t_{\text{val}(v)-1},$$

$$(97) \quad w(e) = 1.$$

The free energy  $F = \log Z$  is given by

$$(98) \quad F = \sum_{\Gamma \in \mathcal{G}^c} \frac{1}{|\text{Aut}(\Gamma)|} \prod_{v \in V(\Gamma)} \lambda^{\text{val}(v)-2} t_{\text{val}(v)-1},$$

where the sum is taken over the set  $\mathcal{G}^c$  all possible connected graphs. Write

$$(99) \quad F = \sum_{g \geq 0} \lambda^{2g-2} F_g.$$



Then

$$(100) \quad F_0 = \sum_{\Gamma \text{ is a tree}} \frac{1}{|\text{Aut}(\Gamma)|} \prod_{v \in V(\Gamma)} t_{\text{val}(v)-1}.$$

The first few terms of  $Z$  are

$$\begin{aligned} Z = & 1 + \left( \frac{\lambda^{-2}}{2} t_0^2 + \frac{1}{2} t_1 \right) + 3 \cdot \left( \frac{t_0^4}{24} \lambda^{-4} + \frac{t_0^2 t_1}{4} \lambda^{-2} + \frac{t_1^2}{8} + \frac{t_0 t_2}{6} + \frac{t_3}{24} \lambda^2 \right) \\ & + 15 \cdot \left( \frac{t_0^6}{720} \lambda^{-6} + \frac{t_1 t_0^4}{48} \lambda^{-4} + \frac{t_0^2 t_1^2}{16} \lambda^{-2} + \frac{t_0^3 t_2}{36} \lambda^{-2} + \frac{t_1^3}{48} + \frac{t_0 t_1 t_2}{12} \right. \\ & + \frac{t_0^2 t_3}{48} + \frac{t_0 t_4}{120} \lambda^2 + \frac{t_1 t_3}{48} \lambda^2 + \frac{t_2^2}{72} \lambda^2 + \frac{t_5}{720} \lambda^4 \Big) \\ & + 105 \left( \frac{t_0^8}{40320} \lambda^{-8} + \frac{t_1 t_0^6}{1440} \lambda^{-6} + \frac{t_2 t_0^5}{720} \lambda^{-4} + \frac{t_1^2 t_0^4}{192} \lambda^{-4} \right. \\ & + \frac{t_2 t_1 t_0^3}{72} \lambda^{-2} + \frac{t_3 t_0^4}{576} \lambda^{-2} + \frac{t_1^3 t_0^2}{96} \lambda^{-2} \\ & + \frac{t_4 t_0^3}{720} + \frac{t_2^2 t_0^2}{144} + \frac{t_1^4}{384} + \frac{t_3 t_1 t_0^2}{96} + \frac{t_2 t_1^2 t_0}{48} \\ & + \frac{t_5 t_0^2}{1440} \lambda^2 + \frac{t_3 t_1^2}{192} \lambda^2 + \frac{t_2^2 t_1}{144} \lambda^2 + \frac{t_4 t_0 t_1}{240} \lambda^2 + \frac{t_3 t_0 t_2}{144} \lambda^2 \\ & + \frac{t_3^2}{1152} \lambda^4 + \frac{t_5 t_1}{1440} \lambda^4 + \frac{t_4 t_2}{720} \lambda^4 + \frac{t_0 t_6}{5040} \lambda^4 + \frac{t_7}{40320} \lambda^6 \Big) + \dots, \end{aligned}$$

and the first few terms of the free energy are given by:

$$\begin{aligned} F = & \left( \frac{1}{2} \lambda^{-2} t_0^2 + \frac{1}{2} t_1 \right) + \left( \frac{1}{2} t_0^2 t_1 \lambda^{-2} + \frac{1}{2} t_0 t_2 + \frac{1}{4} t_1^2 + \frac{1}{8} t_3 \lambda^2 \right) \\ & + \left( \frac{1}{2} t_0^2 t_1^2 \lambda^{-2} + \frac{1}{6} t_0^3 t_2 \lambda^{-2} + \frac{1}{6} t_1^3 + \frac{1}{4} t_0^2 t_3 + t_0 t_2 t_1 \right. \\ & + \frac{5}{24} t_2^2 \lambda^2 + \frac{1}{8} t_0 t_4 \lambda^2 + \frac{1}{4} t_1 t_3 \lambda^2 + \frac{1}{48} t_5 \lambda^4 \Big) \\ & + \left( \frac{1}{2} \lambda^{-2} t_1^3 t_0^2 + \frac{1}{2} \lambda^{-2} t_1 t_2 t_0^3 + \frac{1}{24} \lambda^{-2} t_3 t_0^4 \right. \\ & + \frac{1}{8} t_1^4 + \frac{3}{4} t_0^2 t_1 t_3 + \frac{3}{2} t_0 t_1^2 t_2 + \frac{1}{12} t_0^3 t_4 + \frac{1}{2} t_0^2 t_2^2 \\ & + \frac{5}{8} \lambda^2 t_2^2 t_1 + \frac{3}{8} \lambda^2 t_4 t_0 t_1 + \frac{3}{8} \lambda^2 t_3 t_1^2 + \frac{1}{16} \lambda^2 t_5 t_0^2 + \frac{2}{3} \lambda^2 t_3 t_0 t_2 \\ & + \frac{1}{16} \lambda^4 t_5 t_1 + \frac{1}{12} \lambda^4 t_3^2 + \frac{1}{48} \lambda^4 t_0 t_6 + \frac{7}{48} \lambda^4 t_4 t_2 + \frac{1}{384} t_7 \lambda^6 \Big) + \dots. \end{aligned}$$

Feynman diagrams with no loops are exactly the same as those for  $I_{-1} - 1 - \frac{1}{2} I_0^2$  and the Feynman rules are modified by a power of  $\lambda$  (cf.

Theorem 3.6). The following are some Feynman diagrams with loops:

$$\begin{array}{cccc}
 \text{---}\bigcirc\text{---} & \text{---}\bigcirc\text{---} & \bigcirc\text{---}\bigcirc & \bigcirc\text{---}\bigcirc \\
 \frac{1}{2}t_1 & \frac{1}{2}t_0t_2 & \frac{1}{4}t_1^2 & \frac{1}{8}t_3
 \end{array}$$

If one regards the vertices as atoms and the edges as chemical bonds, then a Feynman diagram corresponds to a polymer (possibly with self bonds). One can also consider the dual diagrams, they are branching chains [8].

We notice that the free energy has the following structure:

$$\begin{aligned}
 F &= \frac{1}{\lambda^2} \left( \frac{1}{2} \frac{t_0^2}{1-t_1} + \frac{1}{6} \frac{t_0^3 t_2}{(1-t_1)^3} + \frac{1}{24} \frac{t_3 t_0^4}{(1-t_1)^4} + \dots \right) \\
 &+ \left( \frac{1}{2} \log \frac{1}{1-t_1} + \frac{1}{2} \frac{t_0 t_2}{(1-t_1)^2} + \frac{1}{4} \frac{t_0^2 t_3}{(1-t_1)^3} + \dots \right) \\
 &+ \lambda^2 \left( \frac{1}{8} \frac{t_3}{(1-t_1)^2} + \frac{1}{8} \frac{t_0 t_4}{(1-t_1)^3} + \frac{5}{24} \frac{t_2^2}{(1-t_1)^3} + \dots \right) + \dots
 \end{aligned}$$

In fact we have

**Theorem 4.3.** *The free energy of topological 1D gravity can be rewritten in the following form:*

$$(101) \quad F = \frac{1}{2} \log(1-t_1) + \sum_{\substack{g,n \geq 0 \\ 2g-2+n > 0}} \sum_{\substack{a_2, \dots, a_n \neq 1 \\ \sum a_j = 2g-2+n}} \frac{\langle \prod_{j=1}^n \tau_{a_j} \rangle_g}{(1-t_1)^{g-1+n}} \prod_{j=1}^n t_{a_j}.$$

This can be easily proved by performing a scaling of variable to get:

$$(102) \quad Z = \frac{(1-t_1)^{1/2}}{\sqrt{2\pi\lambda}} \int_{\mathbb{R}} dx \exp \frac{1}{\lambda^2} \left( -\frac{1}{2} x^2 + \sum_{\substack{n \geq 1 \\ n \neq 2}} \frac{t_{n-1}}{(1-t_1)^{n/2}} \frac{x^n}{n!} \right).$$

The free energy  $F$  now is a summation over all connected graphs without vertices of valence 2, and the propagator is changed to

$$\frac{w_e}{1-w_v} = \frac{1}{1-t_1}.$$

I.e., one can take all the original Feynman diagrams and ignore all the vertices of valence 2. This will produce all Feynman diagrams for the new Feynman rules, except for the following cases:

$$\begin{array}{ccccccccc}
 \bigcirc & = & \bigcirc & + & \bigcirc & + & \bigcirc & + & \bigcirc & + & \dots \\
 \frac{1}{2} \log \frac{1}{1-t_1} & & \frac{1}{2} t_1 & & \frac{1}{4} t_1^2 & & \frac{1}{12} t_1^3 & & \frac{1}{48} t_1^4 & & 
 \end{array}$$

**4.3. Correlators.** The coefficients of  $F$  gives us the correlators defined by:

$$(103) \quad \langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g = \frac{\partial^n}{\partial t_{a_1} \cdots \partial t_{a_n}} F_g|_{\mathbf{t}=0}.$$

These are Taylor series coefficients of  $F$ , in other words,

$$F_g = \sum_{m_0, \dots, m_n \geq 0} \langle \tau_0^{m_0} \cdots \tau_n^{m_n} \rangle_g \prod_{j=0}^n \frac{t_j}{m_j!},$$

$$F = \sum_{g \geq 0} \lambda^{2g-2} F_g.$$

The following are some examples:

$$\begin{aligned} \langle \tau_0^2 \rangle_0 &= 1, \langle \tau_1 \rangle_1 = \frac{1}{2}, \langle \tau_0^2 \tau_1 \rangle_0 = 1, \langle \tau_0 \tau_2 \rangle_1 = \frac{1}{2}, \langle \tau_1^2 \rangle_1 = \frac{1}{2}, \\ \langle \tau_3 \rangle_2 &= \frac{1}{8}, \langle \tau_0^2 \tau_1^2 \rangle_0 = 2, \langle \tau_0^3 \tau_2 \rangle_0 = 1, \langle \tau_1^3 \rangle_1 = 1, \langle \tau_0^2 \tau_3 \rangle_1 = \frac{1}{2}, \\ \langle \tau_0 \tau_1 \tau_2 \rangle_1 &= 1, \langle \tau_2^2 \rangle_2 = \frac{5}{12}, \langle \tau_0 \tau_4 \rangle_2 = \frac{1}{8}, \langle \tau_1 \tau_3 \rangle_2 = \frac{1}{4}, \\ \langle \tau_5 \rangle_3 &= \frac{1}{48}, \langle \tau_1^3 \tau_0^2 \rangle_0 = 6, \langle \tau_1 \tau_2 \tau_0^3 \rangle_0 = 3, \langle \tau_3 \tau_0^4 \rangle_0 = 1, \\ \langle \tau_1^4 \rangle_1 &= 3, \langle \tau_0^2 \tau_1 \tau_3 \rangle_1 = \frac{3}{2}, \langle \tau_0 \tau_1^2 \tau_2 \rangle_1 = 3, \langle \tau_0^3 \tau_4 \rangle_1 = \frac{1}{2}, \\ \langle t_0^2 t_2^2 \rangle_1 &= 2, \langle \tau_2^2 \tau_1 \rangle_2 = \frac{5}{4}, \langle \tau_4 \tau_0 \tau_1 \rangle_2 = \frac{3}{8}, \langle \tau_3 \tau_1^2 \rangle_2 = \frac{3}{4}, \\ \langle \tau_5 \tau_0^2 \rangle_2 &= \frac{1}{8}, \langle \tau_0 \tau_2 \tau_3 \rangle_2 = \frac{2}{3}, \langle \tau_5 \tau_1 \rangle_3 = \frac{1}{16}, \langle t_3^2 \rangle_3 = \frac{1}{6}, \\ \langle \tau_0 \tau_6 \rangle_3 &= \frac{1}{48}, \langle t_4 t_2 \rangle_3 = \frac{7}{48}, \langle \tau_7 \rangle_4 = \frac{1}{384}. \end{aligned}$$

By comparing with (98), we get

**Proposition 4.4.** *The correlators of topological 1D gravity can be given by Feynman sum as follows:*

$$(104) \quad \langle \tau_0^{m_0} \cdots \tau_n^{m_n} \rangle_g = \sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} \cdot \prod_{j=0}^n m_j!,$$

where the summation is taken over connected graphs with  $m_j$  vertices of valence  $j+1$ ,  $j = 0, \dots, n$ .

One can rewrite (98) in terms of correlators as follows:

$$(105) \quad \sum \langle \tau_0^{m_0} \cdots \tau_n^{m_n} \rangle_g \prod_{j=0}^n \frac{t_j^{m_j}}{m_j!} = \sum_{\Gamma \in \mathcal{G}^c} \frac{1}{|\text{Aut}(\Gamma)|} \prod_{v \in V(\Gamma)} \lambda^{\text{val}(v)-2} t_{\text{val}(v)-1}.$$

This is the analogue of Kontsevich's main identity [7].

**4.4. Some special cases of  $Z$  and  $F$ .** In the above we have expressed the partition function and free energy of topological 1D gravity as summation over graphs. By  $Z(t_{a_1}, \dots, t_{a_n})$  we mean taking all  $t_i$ ' to be zero except for  $t_{a_1}, \dots, t_{a_n}$ . We have:

$$(106) \quad Z(t_0) = \exp\left(\frac{t_0^2}{2\lambda^2}\right).$$

In fact,

$$\begin{aligned} Z(t_0) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \exp\left(-\frac{1}{2}x^2 + \frac{t_0}{\lambda^2}x\right) \\ &= \exp\left(\frac{t_0^2}{2\lambda^2}\right) \cdot \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \exp\left(-\frac{1}{2}\left(x - \frac{t_0}{\lambda^2}\right)^2\right) \\ &= \exp\left(\frac{t_0^2}{2\lambda^2}\right) \cdot \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \exp\left(-\frac{1}{2}x^2\right) \\ &= \exp\left(\frac{t_0^2}{2\lambda^2}\right). \end{aligned}$$

Similarly, we also have

$$(107) \quad Z(t_0, t_1) = \exp\left(\frac{t_0^2}{2\lambda^2(1-t_1)} + \frac{1}{2} \log \frac{1}{1-t_1}\right),$$

and in particular,

$$(108) \quad Z(t_1) = \exp\left(\frac{1}{2} \log \frac{1}{1-t_1}\right),$$

by the following computations:

$$\begin{aligned}
Z(t_0, t_1) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \exp\left(-\frac{1}{2}x^2 + \frac{t_0}{\lambda}x + t_1 \frac{x^2}{2}\right) \\
&= \frac{1}{(1-t_1)^{1/2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d((1-t_1)^{1/2}x) \\
&\quad \cdot \exp\left(-\frac{1}{2}((1-t_1)^{1/2}x)^2 - \frac{t_0}{\lambda(1-t_1)^{1/2}}(1-t_1)^{1/2}x\right) \\
&= \exp\left(\frac{t_0^2}{2\lambda^2(1-t_1)} + \frac{1}{2} \log \frac{1}{1-t_1}\right).
\end{aligned}$$

When  $t_n = \delta_{n,2}$  one gets the integral:

$$(109) \quad Z(t_2) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \exp\left(-\frac{1}{2}x^2 + t_2 \lambda \frac{x^3}{3!}\right)$$

Using (86), one gets:

$$(110) \quad Z(t_2) = \sum_{n=0}^{\infty} \frac{t_2^{2n}}{3!^{2n}} \lambda^{2n} \frac{(6n-1)!!}{(2n)!}.$$

The first few terms are

$$\begin{aligned}
Z(t_2) &= 1 + \frac{5}{24}(t_2^2 \lambda^2) + \frac{385}{1152}(t_2^2 \lambda^2)^2 + \frac{85085}{82944}(t_2^2 \lambda^2)^3 + \frac{37182145}{7962624}(t_2^2 \lambda^2)^4 \\
&\quad + \frac{5391411025}{191102976}(t_2^2 \lambda^2)^5 + \frac{5849680962125}{27518828544}(t_2^2 \lambda^2)^6 \\
&\quad + \frac{1267709431363375}{660451885056}(t_2^2 \lambda^2)^7 + \frac{2562040760785380875}{126806761930752}(t_2^2 \lambda^2)^8 + \dots
\end{aligned}$$

After taking logarithm:

$$\begin{aligned}
F(t_2) &= \frac{5}{24}t_2^2 \lambda^2 + \frac{5}{16}(t_2^2 \lambda^2)^2 + \frac{1105}{1152}(t_2^2 \lambda^2)^3 + \frac{565}{128}(t_2^2 \lambda^2)^4 + \frac{82825}{3072}(t_2^2 \lambda^2)^5 \\
&\quad + \frac{19675}{96}(t_2^2 \lambda^2)^6 + \frac{1282031525}{688128}(t_2^2 \lambda^2)^7 + \frac{80727925}{4096}(t_2^2 \lambda^2)^8 + \dots
\end{aligned}$$

In particular,  $F_g(t_2)$  has the following form:

$$(111) \quad F_g(t_2) = b_g t_2^{2g-2}$$

for some constant  $b_g$  for  $g > 1$ .

To compute

$$Z(t_0, t_2) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \exp\left(-\frac{1}{2}x^2 + \frac{t_0}{\lambda}x + \lambda t_2 \frac{x^3}{6}\right),$$

we make a change of variables  $x = y + a$ :

$$\begin{aligned} Z(t_0, t_2) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \exp\left(-\frac{1}{2}(y+a)^2 + \frac{t_0}{\lambda}(y+a) + t_2\lambda \frac{(y+a)^3}{6}\right) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \exp\left(-\frac{1}{2}(1-at_2\lambda)y^2 + (-a + \frac{t_0}{\lambda} + \frac{t_2}{2}\lambda a^2)y \right. \\ &\quad \left. + \frac{t_2}{6}\lambda y^3\right) \cdot \exp\left(-\frac{1}{2}a^2 + \frac{t_0}{\lambda}a + \lambda t_2 \frac{a^3}{6}\right). \end{aligned}$$

We take

$$(112) \quad a = \frac{1 - \sqrt{1 - 2t_0t_2}}{t_2\lambda}$$

such that

$$-a + \frac{t_0}{\lambda} + \frac{t_2}{2}\lambda a^2 = 0,$$

and so we can reduce to the Airy integral and get:

$$\begin{aligned} Z(t_0, t_2) &= \exp\left(\frac{1}{3t_2^2\lambda^2} \left((1-2t_0t_2)^{3/2} - (1-3t_0t_2)\right) + \frac{1}{4} \log(1-2t_0t_2)\right) \\ &\quad \cdot \sum_{n=0}^{\infty} \left(\frac{t_2\lambda}{3!(1-2t_0t_2)^{3/4}}\right)^{2n} \frac{(6n-1)!!}{(2n)!}. \end{aligned}$$

After taking logarithm, one gets:

$$\begin{aligned} F(t_0, t_2) &= \frac{t_0^2}{\lambda^2} \frac{(1-2t_0t_2)^{3/2} - (1-3t_0t_2)}{3(t_0t_2)^2} + \frac{1}{4} \log \frac{1}{1-2t_0t_2} \\ &+ \frac{5}{24} \frac{t_2^2\lambda^2}{(1-2t_0t_2)^{3/2}} + \frac{5}{16} \frac{t_2^4\lambda^4}{(1-2t_0t_2)^3} + \frac{1105}{1152} \frac{t_2^6\lambda^6}{(1-2t_0t_2)^{9/2}} \\ &+ \frac{565}{128} \frac{t_2^8\lambda^8}{(1-2t_0t_2)^6} + \frac{82825}{3072} \frac{t_2^{10}\lambda^{10}}{(1-2t_0t_2)^{15/2}} + \dots \end{aligned}$$

In particular, when  $g > 1$ ,  $F_g(t_0, t_2)$  has the following form:

$$(113) \quad F_g(t_0, t_2) = b_g \frac{t_2^{2g-2}}{(1-2t_0t_2)^{(3g-3)/2}}.$$

In the same fashion as in the case of  $Z(t_2)$ ,

$$(114) \quad Z(t_{2k-1}) = \sum_{n \geq 0} \left(\frac{t_{2k-1}\lambda^{2k-2}}{(2k)!}\right)^n \frac{(2nk-1)!!}{n!},$$

and

$$(115) \quad Z(t_{2k}) = \sum_{n \geq 0} \left(\frac{t_{2k}\lambda^{2k-1}}{(2k+1)!}\right)^{2n} \frac{(2n(2k+1)-1)!!}{(2n)!}.$$

**4.5. General explicit expression for  $Z$ .** One can generalize the formulas in the preceding subsection for  $Z(t_n)$  as follows:

**Proposition 4.5.** *The partition function  $Z$  has the following closed expression:*

$$(116) \quad Z = \sum_{n \geq 0} \sum_{\sum_{j=1}^k m_j j = 2n} \frac{(2n-1)!!}{\prod_{j=1}^k (j!)^{m_j} m_j!} \lambda^{2n-2 \sum_{j=1}^k m_j} \cdot \prod_{j=1}^k t_{j-1}^{m_j}.$$

*Proof.*

$$\begin{aligned} Z &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \exp\left(-\frac{1}{2}x^2\right) \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{t_{n-1}}{n!} \lambda^{n-2} x^n\right)^l \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \exp\left(-\frac{1}{2}x^2\right) \sum_{m_1, \dots, m_n=0}^{\infty} \prod_{j=1}^n \frac{1}{m_j!} \left(\frac{t_{j-1}}{j!} \lambda^{j-2}\right)^{m_j} x^{\sum_{j=1}^n m_j j} \\ &= \sum_{n \geq 0} \sum_{\sum_{j=1}^k m_j j = 2n} \frac{(2n-1)!!}{\prod_{j=1}^k (j!)^{m_j} m_j!} \lambda^{2n-2 \sum_{j=1}^k m_j} \cdot \prod_{j=1}^k t_{j-1}^{m_j}. \end{aligned}$$

□

By comparing (116) with (95), one gets

**Corollary 4.6.** *The following Feynman sum has a close form expression:*

$$\begin{aligned} &\sum_{\Gamma \in \mathcal{G}} \frac{1}{|\text{Aut}(\Gamma)|} \prod_{v \in V(\Gamma)} \lambda^{\text{val}(v)-2} t_{\text{val}(v)-1} \\ &= \sum_{n \geq 0} \sum_{\sum_{j=1}^k m_j j = 2n} \frac{(2n-1)!!}{\prod_{j=1}^k (j!)^{m_j} m_j!} \lambda^{2n-2 \sum_{j=1}^k m_j} \cdot \prod_{j=1}^k t_{j-1}^{m_j}. \end{aligned}$$

**Proposition 4.7.** *The partition function  $Z$  has the following closed expression:*

$$(117) \quad \begin{aligned} Z &= \frac{1}{\sqrt{1-t_1}} \sum_{n \geq 0} \sum_{\substack{\sum_{j=1}^k m_j j = 2n \\ m_2=0}} \frac{(2n-1)!!}{\prod_{j=1}^k (j!)^{m_j} m_j!} \lambda^{2n-2 \sum_{j=1}^k m_j} \\ &\quad \cdot \prod_{j=1}^k \left( \frac{t_{j-1}}{(1-t_1)^{j/2}} \right)^{m_j}. \end{aligned}$$

*Proof.*

$$\begin{aligned}
Z &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \exp\left(-\frac{1}{2}(1-t_1)x^2\right) \exp \sum_{n \geq 1, \neq 2} \frac{t_{n-1}}{n!} \lambda^{n-2} x^n \\
&= \frac{1}{\sqrt{2\pi(1-t_1)}} \int_{\mathbb{R}} dx \exp\left(-\frac{1}{2}x^2\right) \exp \sum_{n \geq 1, \neq 2} \frac{t_{n-1} \lambda^{n-2} x^n}{n!(1-t_1)^{n/2}} \\
&= \frac{1}{\sqrt{1-t_1}} \sum_{n \geq 0} \sum_{\substack{\sum_{j=1}^k m_j = 2n \\ m_2 = 0}} \frac{(2n-1)!!}{\prod_{j=1}^k (j!)^{m_j} m_j!} \lambda^{2n-2 \sum_{j=1}^k m_j} \\
&\quad \cdot \prod_{j=1}^k \left( \frac{t_{j-1}}{(1-t_1)^{j/2}} \right)^{m_j}.
\end{aligned}$$

□

The following are the first few terms:

$$\begin{aligned}
Z &= 1 + \left( \frac{1}{2} \lambda^{-2} t_0^2 + \frac{1}{2} t_1 \right) + \left( \frac{1}{8} \lambda^{-4} t_0^4 + \frac{3}{4} \lambda^{-2} t_0^2 t_1 + \frac{3}{8} t_1^2 + \frac{1}{2} t_0 t_2 + \frac{1}{8} \lambda^2 t_3 \right) \\
&+ \left( \frac{1}{48} \lambda^{-6} t_0^6 + \frac{5}{16} \lambda^{-4} t_0^4 t_1 + \frac{15}{16} \lambda^{-2} t_0^2 t_1^2 + \frac{5}{16} t_1^3 + \frac{5}{12} \lambda^{-2} t_0^3 t_2 \right. \\
&+ \left. \frac{5}{4} t_0 t_1 t_2 + \frac{5}{24} \lambda^2 t_2^2 + \frac{5}{16} t_0^2 t_3 + \frac{5}{16} \lambda^2 t_1 t_3 + \frac{1}{8} \lambda^2 t_0 t_4 + \frac{1}{48} \lambda^4 t_5 \right) + \dots
\end{aligned}$$

$$\begin{aligned}
Z &= \frac{1}{\sqrt{1-t_1}} \left( 1 + \frac{1}{2} \lambda^{-2} \frac{t_0^2}{1-t_1} + \frac{1}{(1-t_1)^2} \left( \frac{1}{8} \lambda^{-4} t_0^4 + \frac{1}{2} t_0 t_2 + \frac{1}{8} \lambda^2 t_3 \right) \right. \\
&+ \frac{1}{(1-t_1)^3} \left( \frac{1}{48} \lambda^{-6} t_0^6 + \frac{5}{12} \lambda^{-2} t_0^3 t_2 \right. \\
&+ \left. \left. \frac{5}{24} \lambda^2 t_2^2 + \frac{5}{16} t_0^2 t_3 + \frac{1}{8} \lambda^2 t_0 t_4 + \frac{1}{48} \lambda^4 t_5 \right) + \dots \right)
\end{aligned}$$



$$\begin{aligned}
F = & \left( \frac{1}{2} \lambda^{-2} t_0^2 + \frac{1}{2} t_1 \right) + \left( \frac{1}{2} t_0^2 t_1 \lambda^{-2} + \frac{1}{2} t_0 t_2 + \frac{1}{4} t_1^2 + \frac{1}{8} t_3 \lambda^2 \right) \\
& + \left( \frac{1}{2} t_0^2 t_1^2 \lambda^{-2} + \frac{1}{6} t_0^3 t_2 \lambda^{-2} + \frac{1}{6} t_1^3 + \frac{1}{4} t_0^2 t_3 + t_0 t_2 t_1 \right) \\
& + \left( \frac{5}{24} t_2^2 \lambda^2 + \frac{1}{8} t_0 t_4 \lambda^2 + \frac{1}{4} t_1 t_3 \lambda^2 + \frac{1}{48} t_5 \lambda^4 \right) \\
& + \left( \frac{1}{2} \lambda^{-2} t_1^3 t_0^2 + \frac{1}{2} \lambda^{-2} t_1 t_2 t_0^3 + \frac{1}{24} \lambda^{-2} t_3 t_0^4 \right) \\
& + \frac{1}{8} t_1^4 + \frac{3}{4} t_0^2 t_1 t_3 + \frac{3}{2} t_0 t_1^2 t_2 + \frac{1}{12} t_0^3 t_4 + \frac{1}{2} t_0^2 t_2^2 \\
& + \frac{5}{8} \lambda^2 t_2^2 t_1 + \frac{3}{8} \lambda^2 t_4 t_0 t_1 + \frac{3}{8} \lambda^2 t_3 t_1^2 + \frac{1}{16} \lambda^2 t_5 t_0^2 + \frac{2}{3} \lambda^2 t_3 t_0 t_2 \\
& + \frac{1}{16} \lambda^4 t_5 t_1 + \frac{1}{12} \lambda^4 t_3^2 + \frac{1}{48} t_0 \lambda^4 t_6 + \frac{7}{48} \lambda^4 t_4 t_2 + \frac{1}{384} t_7 \lambda^6 + \dots,
\end{aligned}$$

The coefficients of  $F$  give us correlators. For example,

$$\begin{aligned}
\langle \tau_3 \rangle_2 &= \frac{1}{8}, \quad \langle \tau_2^2 \rangle_2 = \frac{5}{12}, \\
\langle \tau_5 \rangle_3 &= \frac{1}{48}, \quad \langle \tau_3^2 \rangle_3 = \frac{1}{6}, \quad \langle \tau_2 \tau_4 \rangle_3 = \frac{7}{48}, \quad \langle \tau_7 \rangle_4 = \frac{1}{384}.
\end{aligned}$$

We also have:

$$\begin{aligned}
F = & \frac{1}{2} \log \frac{1}{1-t_1} + \frac{1}{2} \lambda^{-2} \frac{t_0^2}{1-t_1} + \frac{1}{(1-t_1)^2} \left( \frac{1}{2} t_0 t_2 + \frac{1}{8} t_3 \lambda^2 \right) \\
& + \frac{1}{(1-t_1)^3} \left( \frac{1}{6} t_0^3 t_2 \lambda^{-2} + \frac{1}{4} t_0^2 t_3 + \frac{5}{24} t_2^2 \lambda^2 + \frac{1}{8} t_0 t_4 \lambda^2 + \frac{1}{48} t_5 \lambda^4 \right) \\
& + \frac{1}{(1-t_1)^4} \left( \frac{1}{24} \lambda^{-2} t_3 t_0^4 + \frac{1}{12} t_0^3 t_4 + \frac{1}{2} t_0^2 t_2^2 + \frac{1}{16} \lambda^2 t_5 t_0^2 \right) \\
& + \frac{2}{3} \lambda^2 t_3 t_0 t_2 + \frac{1}{12} \lambda^4 t_3^2 + \frac{1}{48} t_0 \lambda^4 t_6 + \frac{7}{48} \lambda^4 t_4 t_2 + \frac{1}{384} t_7 \lambda^6 + \dots.
\end{aligned}$$

**4.6. The selection rule.** A nonzero term in  $Z$  is of the form

$$(118) \quad t_{a_1} \cdots t_{a_n} \lambda^{2g-2},$$

up to coefficients. The numbers  $a_1 + 1, \dots, a_n + 1$  gives a partition  $\mu$  of length  $n$ , so by (116),

$$2g - 2 = \sum_{j=1}^n (a_j + 1) - 2n,$$

so one must have

$$(119) \quad a_1 + \cdots + a_n = 2g - 2 + n.$$

This is the *selection rule* for nonvanishing terms in  $Z$ . After taking logarithm, one gets the same rule for  $F$ .

As an application of the selection rule, we have

**Proposition 4.8.** *The genus zero part  $F_0$  of the free energy satisfies the following initial condition:*

$$(120) \quad F_0|_{t_0=0} = 0.$$

*Proof.* Just take  $g = 0$  in (119).  $\square$

For  $g \geq 1$ , we do not have  $F_g|_{t_0=0} = 0$ .

Another application of the selection rule is the following

**Proposition 4.9.** *The free energy  $F$  restricted to the  $t_0$ -line is given by:*

$$(121) \quad F(t_0) = \frac{\lambda^{-2}}{2} t_0^2.$$

*Proof.* Take  $a_1 = \cdots = a_n = 0$  in (119) to get:

$$2g + n = 2.$$

Then one has  $g = 0$  and  $n = 2$ .  $\square$

**4.7. Partition function and free energy in I-coordinates.** Recall by Theorem 2.5,

$$(122) \quad \begin{aligned} S(x) = & \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1} \\ & - \frac{1}{2} (1 - I_1) (x - I_0)^2 + \sum_{n=3}^{\infty} \frac{I_{n-1}}{n!} (x - I_0)^n, \end{aligned}$$

therefore, by (88),

$$(123) \quad \begin{aligned} Z = & \exp \left( \frac{1}{\lambda^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1} \right) \\ & \cdot \frac{1}{\sqrt{\pi}\lambda} \int dx \exp \frac{1}{\lambda^2} \left( -\frac{1}{2} (1 - I_1) x^2 + \sum_{n \geq 3} I_{n-1} \frac{x^n}{n!} \right). \end{aligned}$$

After a scaling of the variable  $x$ ,

$$(124) \quad \begin{aligned} Z = & \exp \left( \frac{1}{\lambda^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1} + \frac{1}{2} \log \frac{1}{1 - I_1} \right) \\ & \cdot \frac{1}{\sqrt{\pi}\lambda} \int dx \exp \frac{1}{\lambda^2} \left( -\frac{1}{2} x^2 + \sum_{n \geq 3} \frac{I_{n-1}}{(1 - I_1)^{n/2}} \frac{x^n}{n!} \right). \end{aligned}$$

Therefore, by (117), we obtain the following:

**Theorem 4.10.** *The partition function of the topological 1D gravity can be expressed in  $I$ -coordinates as follows:*

$$(125) \quad Z = \exp\left(\frac{1}{\lambda^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1} + \frac{1}{2} \log \frac{1}{1-I_1}\right) \\ \cdot \sum_{n \geq 0} \sum_{\substack{k \\ \sum_{j=3}^k m_j j = 2n}} \frac{(2n-1)!!}{\prod_{j=3}^k (j!)^{m_j} m_j!} \lambda^{2n-2 \sum_{j=1}^k m_j} \cdot \prod_{j=3}^k \left(\frac{I_{j-1}}{(1-I_1)^{j/2}}\right)^{m_j}.$$

For example,

$$Z = \exp\left(\frac{1}{\lambda^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1} + \frac{1}{2} \log \frac{1}{1-I_1}\right) \\ \cdot \left(1 + \frac{1}{8} \frac{I_3}{(1-I_1)^2} \lambda^2 + \frac{1}{(1-I_1)^3} \left(\frac{5}{24} I_2^2 \lambda^2 + \frac{1}{48} I_5 \lambda^4\right) \right. \\ \left. + \frac{1}{(1-I_1)^4} \left(\frac{35}{384} I_3^2 \lambda^4 + \frac{7}{48} I_2 I_4 \lambda^4 + \frac{1}{384} I_7 \lambda^6\right) + \dots\right)$$

so after taking logarithm:

$$F = \frac{1}{\lambda^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1} + \frac{1}{2} \log \frac{1}{1-I_1} \\ + \frac{1}{8} \frac{I_3}{(1-I_1)^2} \lambda^2 + \frac{1}{(1-I_1)^3} \left(\frac{5}{24} I_2^2 \lambda^2 + \frac{1}{48} I_5 \lambda^4\right) \\ + \frac{1}{(1-I_1)^4} \left(\frac{1}{12} I_3^2 \lambda^4 + \frac{7}{48} I_4 I_2 \lambda^4 + \frac{1}{384} I_7 \lambda^6\right) + \dots.$$

In particular,

$$(126) \quad F_0 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1},$$

$$(127) \quad F_1 = \frac{1}{2} \log \frac{1}{1-I_1},$$

$$(128) \quad F_2 = \frac{1}{8} \frac{I_3}{(1-I_1)^2} + \frac{5}{24} \frac{I_2^2}{(1-I_1)^3}.$$

In general, for  $g \geq 2$ ,

$$(129) \quad F_g = \sum_{\sum_{j \geq 3} m_j(j-2) = 2g-2} \langle \prod_{j \geq 3} \tau_{j-1}^{m_j} \rangle_g \cdot \prod_{j \geq 3} \frac{1}{m_j!} \left(\frac{I_{j-1}}{(1-I_1)^{j/2}}\right)^{m_j}.$$

We will rederive such formulas in later Sections by different methods.

**4.8. Feynman rules for free energy in  $I$ -coordinates.** By (124) one gets the following Feynman rules for  $F_g$  ( $g \geq 2$ ):

$$(130) \quad F_g = \sum_{\Gamma \in \mathcal{G}_g^c} \frac{1}{|\text{Aut}(\Gamma)|} \prod_{v \in V(\Gamma)} I_{\text{val}(v)-1} \cdot \prod_{e \in E(\Gamma)} \frac{1}{1 - I_1},$$

where the summation is taken over the set of  $g$ -loop connected graphs whose vertices all have valences  $\geq 3$ . For example, in the case of  $F_2$ , we have the following three diagrams:

$$\begin{array}{ccc} \text{Diagram 1: Two vertices connected by two edges} & \text{Diagram 2: Two vertices connected by three edges} & \text{Diagram 3: Two vertices connected by four edges} \\ \frac{1}{12} \frac{I_2^2}{(1-I_1)^3} & \frac{1}{8} \frac{I_2^2}{(1-I_1)^3} & \frac{1}{8} \frac{I_3}{(1-I_1)^2} \end{array}$$

**4.9. Free energy in derivatives of  $I_0$  and the corresponding Feynman rules.** We combine the identities (129) with (63) to get for  $g \geq 2$ ,

$$(131) \quad F_g = \sum_{\sum_{j \geq 3} m_j(j-2) = 2g-2} \langle \prod_{j \geq 3} \tau_{j-1}^{m_j} \rangle_g \cdot \prod_{n \geq 3} \frac{1}{m_n!} \left( - \sum_{\sum_{j \geq 1} j m_j = n-1} \frac{(\sum_j (j+1)n_j)!}{\prod_j ((j+1)!)^{n_j} n_j!} \cdot \frac{\prod_j \left( -\frac{\partial^{j+1} I_0}{\partial t_0^{j+1}} \right)^{n_j}}{\left( \frac{\partial I_0}{\partial t_0} \right)^{\sum_j (j/2+1)n_j}} \right)^{m_n}.$$

This expresses  $F_g$  in terms of derivatives of  $I_0$ . We will later show that

$$I_0 = \frac{\partial F_0}{\partial t_0},$$

so we have expressed  $F_g$  in terms of  $\frac{\partial^2 F_0}{\partial t_0^2}$  and its derivative. This is analogous to similar results in 2D topological gravity where  $F_g$  is expressed in terms of  $\frac{\partial^3 F_0}{\partial t_0^3}$  and its derivative. One can also obtain such expressions by combining (130) with (63). It is an interesting problem to formulate the Feynman rules for such expressions. The following are some examples:

$$F_2 = \frac{1}{8} \frac{\frac{\partial^3 I_0}{\partial t_0^3}}{\left( \frac{\partial I_0}{\partial t_0} \right)^2} - \frac{1}{12} \frac{\left( \frac{\partial^2 I_0}{\partial t_0^2} \right)^2}{\left( \frac{\partial I_0}{\partial t_0} \right)^3}.$$

It is interesting to interpret it as a sum over two-loop diagrams:

$$\begin{array}{ccc}
 \text{Diagram 1} & \text{Diagram 2} & \text{Diagram 3} \\
 \text{---} \frac{1}{12} \frac{(\partial_{t_0}^2 I_0)^2}{(\partial_{t_0} I_0)^3} & 0 \cdot \frac{(\partial_{t_0}^2 I_0)^2}{(\partial_{t_0} I_0)^3} & \frac{1}{8} \frac{\partial_{t_0}^3 I_0}{(\partial_{t_0} I_0)^2}
 \end{array}$$

The factor  $\frac{1}{8}$  is just  $\frac{1}{|\text{Aut}(\Gamma)|}$  for the corresponding diagram  $\Gamma$ , the factor  $-\frac{1}{12}$  is  $(-1)^{|V(\Gamma)-1}(|V(\Gamma)|)! \cdot \frac{1}{|\text{Aut}(\Gamma)|}$ . The factor is 0 for the middle diagram seems to indicate only 1PI diagrams have contributions. Similarly,

$$\begin{aligned}
 F_3 &= \frac{1}{48} \frac{I_5}{(1-I_1)^3} + \frac{1}{12} \frac{I_3^2}{(1-I_1)^4} + \frac{7}{48} \frac{I_4 I_2}{(1-I_1)^4} + \frac{25}{48} \frac{I_2^2 I_3}{(1-I_1)^5} + \frac{5}{16} \frac{I_2^4}{(1-I_1)^6} \\
 &= \frac{(\partial_{t_0}^5 I_0)}{48(\partial_{t_0} I_0)^3} - \frac{(\partial_{t_0}^3 I_0)^2}{8(\partial_{t_0} I_0)^4} - \frac{(\partial_{t_0}^2 I_0)(\partial_{t_0}^4 I_0)}{6(\partial_{t_0} I_0)^4} + \frac{3(\partial_{t_0}^2 I_0)^2(\partial_{t_0}^3 I_0)}{4(\partial_{t_0} I_0)^5} - \frac{1}{2} \frac{(\partial_{t_0} I_0)^4}{(\partial_{t_0} I_0)^6}.
 \end{aligned}$$

We have checked in this case only 1PI programs contributes when one interpret the formula in derivatives of  $I_0$ .

## 5. FLOW EQUATIONS, POLYMER EQUATION, AND THEIR APPLICATIONS

In this Section we present some applications of the flow equation and polymer equation [8] of topological 1D gravity. These include a proof of Theorem 3.1, some explicit expressions of  $F_0$  and its derivatives in  $t_0$  in  $t$ -coordinates and a rederivation of the formula for  $F_0$  in  $I$ -coordinates.

### 5.1. Flow equations.

**Proposition 5.1.** ([8] ) *For each  $n \geq 0$ , the following equation is satisfied by  $Z$ :*

$$(132) \quad \frac{\partial Z}{\partial t_n} = \frac{\lambda^{2n}}{(n+1)!} \frac{\partial^{n+1} Z}{\partial t_0^{n+1}}$$

*Proof.*

$$\begin{aligned}
 \frac{\partial Z}{\partial t_n} &= \frac{1}{\sqrt{2\pi}\lambda} \int_{\mathbb{R}} dx \frac{x^{n+1}}{(n+1)!\lambda^2} \exp \frac{1}{\lambda^2} \left( -\frac{1}{2}x^2 + \sum_{n \geq 1} t_{n-1} \frac{x^n}{n!} \right) \\
 &= \frac{\lambda^{2n}}{(n+1)!} \frac{\partial^{n+1}}{\partial t_0^{n+1}} \frac{1}{\sqrt{2\pi}\lambda} \int_{\mathbb{R}} dx \exp \frac{1}{\lambda^2} \left( -\frac{1}{2}x^2 + \sum_{n \geq 1} t_{n-1} \frac{x^n}{n!} \right) \\
 &= \frac{\lambda^{2n}}{(n+1)!} \frac{\partial^{n+1} Z}{\partial t_0^{n+1}}
 \end{aligned}$$

□

**5.2. Solution of the flow equations.** It is easy to see that the flow equations have the following solution:

$$(133) \quad Z = \exp\left(\sum_{n=1}^{\infty} \frac{\lambda^{2n}}{(n+1)!} t_n \frac{\partial^{n+1}}{\partial t_0^{n+1}}\right) \exp \frac{t_0^2}{2\lambda^2}.$$

This is a sort of free field realization of the topological 1D gravity. One can write  $Z$  as a summation over Feynman diagrams as follows. First expand the two exponentials:

$$(134) \quad Z = \sum_{m_1, \dots, m_n \geq 0} \prod_{j=1}^n \frac{1}{m_j!} \left( \frac{\lambda^{2j}}{(j+1)!} t_j \frac{\partial^{j+1}}{\partial t_0^{j+1}} \right)^{m_j} \sum_{m \geq 0} \frac{1}{m!} \left( \frac{t_0^2}{2\lambda^2} \right)^m.$$

We understand each copy of  $\frac{\partial^{j+1}}{\partial t_0^{j+1}}$  as a vertex marked by  $\bullet$  with  $j+1$  edges, and each copy of  $t_0^2$  as a vertex with  $*$  with two edges. All the edges from a vertex with  $\bullet$  must be connected with a vertex connected a vertex marked with  $*$ , but not vice versa. If an edge from a vertex marked with  $*$  is not connected to another vertex, mark its open end by  $\bullet$ . Then one gets some graphs with vertices marked by  $\bullet$  or  $*$ . Then

$$(135) \quad Z = \sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} w_{\Gamma},$$

where the Feynman rule is where the weight of  $\Gamma$  is given by

$$(136) \quad w_{\Gamma} = \prod_{v \in V(\Gamma)} w_v \cdot \prod_{e \in E(\Gamma)} w_e,$$

with  $w_e$  and  $w_v$  given by the following Feynman rule:

$$(137) \quad w(e) = 1,$$

$$(138) \quad w(v) = \begin{cases} \lambda^{\text{val}(v)} t_{\text{val}(v)-1}, & \text{if } v \text{ is a vertex marked by } \bullet, \\ 1, & \text{if } v \text{ is a vertex marked by } *. \end{cases}$$

Now one can simply ignore the vertices marked by  $*$ . This will not change  $\text{Aut}(\Gamma)$  or  $\Gamma$ , so one gets exactly the Feynman diagrams and Feynman rules as in (??).

As an example, we have

$$\begin{aligned} Z(t_0, t_2) &= \sum_{m_2 \geq 0} \frac{1}{m_2!} \left( \frac{\lambda^4}{6} t_2 \frac{\partial^3}{\partial t_0^3} \right)^{m_2} \sum_{m \geq 0} \frac{1}{m!} \left( \frac{t_0^2}{2\lambda^2} \right)^m \\ &= \sum_{\substack{m_2, m \geq 0 \\ 2m \geq 3m_2}} \frac{1}{m_2!} \left( \frac{\lambda^4}{6} t_2 \right)^{m_2} \frac{1}{m!} \left( \frac{1}{2\lambda^2} \right)^m \prod_{j=0}^{3m_2-1} (2m-j) \cdot t_0^{2m-3m_2}. \end{aligned}$$

In particular,

$$\begin{aligned} Z(t_2) &= \sum_{n \geq 0} \frac{1}{(2n)!} \left( \frac{\lambda^4}{6} t_2 \right)^{2n} \frac{1}{(3n)!} \left( \frac{1}{2\lambda^2} \right)^{3n} (6n)! \\ &= \sum_{n \geq 0} \frac{(6n)! \lambda^{2n}}{288^n (2n)! (3n)!} t_2^{2n}. \end{aligned}$$

The formula (133) is very elegant, but it does not give us any information about the analytic properties of the free energy.

### 5.3. The polymer equation.

**Theorem 5.2.** ([8] ) *The partition function of the topological 1D gravity satisfies the following equation:*

$$(139) \quad \sum_{n \geq 0} \frac{t_n - \delta_{n,1}}{n!} \lambda^{2n} \frac{\partial^n}{\partial t_0^n} Z = 0.$$

*Proof.*

$$\begin{aligned} & \sum_{n \geq 0} \frac{t_n - \delta_{n,1}}{n!} \lambda^{2n} \frac{\partial^n}{\partial t_0^n} Z \\ &= \frac{1}{\sqrt{2\pi\lambda}} \int_{\mathbb{R}} dx \cdot \left( -x + \sum_{n \geq 0} t_n \frac{x^n}{n!} \right) \exp \frac{1}{\lambda^2} \left( -\frac{1}{2} x^2 + \sum_{n \geq 1} t_{n-1} \frac{x^n}{n!} \right) \\ &= \frac{1}{\sqrt{2\pi\lambda}} \int_{\mathbb{R}} dx \cdot \frac{d}{dx} \exp \frac{1}{\lambda^2} \left( -\frac{1}{2} x^2 + \sum_{n \geq 1} t_{n-1} \frac{x^n}{n!} \right) \\ &= 0. \end{aligned}$$

□

In [8, 9], (139) has been called the *polymer equation* or the *chain equation*.

**5.4. Checking the polymer equation.** Let  $t_n = \delta_{n,0}$ , then (139) becomes:

$$(140) \quad \lambda^2 \frac{\partial}{\partial t_0} Z(t_0) = t_0 Z(t_0).$$

This matches with (106).

Let  $t_n = 0$  for  $n > 1$ , then (139) becomes:

$$(141) \quad t_0 Z(t_0, t_1) + (t_1 - 1) \lambda^2 \frac{\partial}{\partial t_0} Z(t_0, t_1) = 0.$$

It can be rewritten as:

$$(142) \quad \frac{\partial}{\partial t_0} F(t_0, t_1) = \frac{t_0}{(1 - t_1)\lambda^2}.$$

This completely determines  $\frac{\partial}{\partial t_0} F(t_0, t_1)$ . After integration:

$$(143) \quad F(t_0, t_1) = \frac{1}{2} \frac{t_0^2}{(1 - t_1)\lambda^2} + F(t_1).$$

So the polymer equation determines  $F(t_0, t_1)$  up to the initial value  $F(t_1)$ .

Let  $t_n = 0$  for  $n = 1$  or  $n > 2$ , then (139) becomes:

$$(144) \quad t_0 Z(t_0, t_2) - \lambda^2 \frac{\partial}{\partial t_0} Z(t_0, t_1) + \lambda^4 \frac{t_2}{2} \frac{\partial^2}{\partial t_0^2} Z(t_0, t_2) = 0.$$

It can be rewritten as:

$$(145) \quad \frac{\partial}{\partial t_0} F(t_0, t_2) = \frac{t_0}{\lambda^2} + \frac{\lambda^2}{2} t_2 ((\partial_{t_0} F(t_0, t_2))^2 + \partial_{t_0}^2 F(t_0, t_2)).$$

After writing  $F(t_0, t_2) = \sum_{g \geq 0} \lambda^{2g-2} F_g(t_0, t_2)$ , one gets

$$\begin{aligned} \partial_{t_0} F_0(t_0, t_2) &= t_0 + \frac{t_2}{2} (\partial_{t_0} F(t_0, t_2))^2, \\ \partial_{t_0} F_g(t_0, t_2) &= \frac{t_2}{2} \left( \sum_{h=0}^g \partial_{t_0} F_h(t_0, t_2) \cdot \partial_{t_0} F_{g-h}(t_0, t_2) + \partial_{t_0}^2 F_{g-1}(t_0, t_2) \right), \end{aligned}$$

for  $g \geq 1$ . From the first equation one can get

$$\partial_{t_0} F_0(t_0, t_2) = \frac{1 - (1 - 2t_0 t_2)^{1/2}}{t_2},$$

and one can rewrite the second equation as

$$\begin{aligned} & \partial_{t_0} F_g(t_0, t_2) \\ &= \frac{t_2}{2(1 - t_2 \partial_{t_0} F_0(t_0, t_2))} \left( \sum_{h=1}^{g-1} \partial_{t_0} F_h(t_0, t_2) \cdot \partial_{t_0} F_{g-h}(t_0, t_2) + \partial_{t_0}^2 F_{g-1}(t_0, t_2) \right) \\ &= \frac{t_2}{2(1 - 2t_0 t_2)^{1/2}} \left( \sum_{h=1}^{g-1} \partial_{t_0} F_h(t_0, t_2) \cdot \partial_{t_0} F_{g-h}(t_0, t_2) + \partial_{t_0}^2 F_{g-1}(t_0, t_2) \right) \end{aligned}$$



so it can be used to find  $\partial_{t_0} F_g(t_0, t_2)$  for all  $g \geq 1$  recursively. For example,

$$\begin{aligned}\partial_{t_0} F_1(t_0, t_2) &= \frac{t_2}{2(1 - 2t_0 t_2)^{1/2}} \partial_{t_0}^2 F_0(t_0, t_2) = \frac{t_2}{2(1 - 2t_0 t_2)}, \\ \partial_{t_0} F_2(t_0, t_2) &= \frac{t_2}{2(1 - 2t_0 t_2)^{1/2}} \left( (\partial_{t_0} F_1(t_0, t_2))^2 + \partial_{t_0}^2 F_1(t_0, t_2) \right) \\ &= \frac{5t_2^3}{8(1 - 2t_0 t_2)^{5/2}}.\end{aligned}$$

For  $g > 1$ , write

$$(146) \quad \frac{\partial F_g}{\partial t_0}(t_0, t_2) = a_g \frac{t_2^{2g-1}}{(1 - 2t_0 t_2)^{(3g-1)/2}},$$

then the above recursion relation yields:

$$(147) \quad a_g = \frac{1}{2} \left( \sum_{h=1}^{g-1} a_h a_{g-h} + (3g-4)a_{g-1} \right), \quad g \geq 2.$$

Define the generating series:

$$(148) \quad A(t) = \sum_{g \geq 1} a_g t^g$$

Then the recursion relation is equivalent to the following differential equation:

$$(149) \quad \frac{3t^2}{2} A'(t) + \frac{1}{2} A(t)^2 - \left(1 + \frac{t}{2}\right) A(t) + \frac{t}{2} = 0.$$

Again the polymer equation only determines  $\frac{\partial F}{\partial t_0}(t_0, t_2)$ . To determine  $F(t_0, t_2)$  we need extra information about the initial value  $F(t_2)$ . We have already shown that

$$(150) \quad F_g(t_0, t_2) = b_g \frac{t_2^{2g-2}}{(1 - 2t_0 t_2)^{(3g-3)/2}}.$$

for some constant  $b_g$  when  $g > 1$ ,

$$(151) \quad F_g(t_0, t_2) = \frac{a_g}{3g-3} \frac{t_2^{2g-2}}{(1 - 2t_0 t_2)^{(3g-3)/2}}.$$

These match with the formula for  $F(t_0, t_2)$  in §4.4.

**5.5. Comparison with the KdV.** Note

$$(152) \quad \frac{\partial^n}{\partial t_0^n} Z = \lambda^{2n} \left( \frac{\partial}{\partial t_0} + \frac{\partial F}{\partial t_0} \right)^n 1 \cdot Z.$$

Therefore, one can rewrite the universal polymer equation as follows:

$$(153) \quad \sum_{n \geq 0} \frac{t_n - \delta_{n,1}}{n!} \lambda^{2n} \left( \frac{\partial}{\partial t_0} + \frac{\partial F}{\partial t_0} \right)^n 1 = 0.$$

Similarly, the flow equation can be written as:

$$(154) \quad \frac{\partial F}{\partial t_n} = \frac{\lambda^{2n}}{(n+1)!} \left( \frac{\partial}{\partial t_0} + \frac{\partial F}{\partial t_0} \right)^{n+1} 1.$$

By taking  $\frac{\partial}{\partial t_0}$  on both sides of (153):

$$(155) \quad 1 + \sum_{n \geq 1} \frac{t_n - \delta_{n,1}}{n!} \lambda^{2n} \frac{\partial}{\partial t_0} \left( \frac{\partial}{\partial t_0} + \frac{\partial F}{\partial t_0} \right)^n 1 = 0.$$

This can be rewritten as:

$$(156) \quad \frac{\partial}{\partial t_n} \frac{\partial F}{\partial t_0} = \frac{\lambda^{2n}}{(n+1)!} \frac{\partial}{\partial t_0} \left( \frac{\partial}{\partial t_0} + \frac{\partial F}{\partial t_0} \right)^{n+1} 1.$$

By induction one can show that

$$(157) \quad \frac{\partial}{\partial t_0} \left( \frac{\partial}{\partial t_0} + \frac{\partial F}{\partial t_0} \right)^n 1 = \left[ \left( \frac{\partial}{\partial t_0} + \frac{\partial F}{\partial t_0} \right)^n, \frac{\partial F}{\partial t_0} \right] 1.$$

Therefore, one has

$$(158) \quad 1 + \sum_{n \geq 1} \frac{t_n - \delta_{n,1}}{n!} \lambda^{2n} \left[ \left( \frac{\partial}{\partial t_0} + \frac{\partial F}{\partial t_0} \right)^n, \frac{\partial F}{\partial t_0} \right] 1 = 0,$$

and

$$(159) \quad \frac{\partial}{\partial t_n} \frac{\partial F}{\partial t_0} = \frac{\lambda^{2n}}{(n+1)!} \left[ \left( \frac{\partial}{\partial t_0} + \frac{\partial F}{\partial t_0} \right)^{n+1}, \frac{\partial F}{\partial t_0} \right] 1.$$

These formulas were derived in [9].

**5.6. Explicit formula for  $\frac{\partial F_0}{\partial t_0}$ .** By (153), we have

$$(160) \quad \frac{\partial F_0}{\partial t_0} = \sum_{n \geq 0} \frac{t_n}{n!} \left( \frac{\partial F_0}{\partial t_0} \right)^n.$$

This is exactly the equation (18) satisfied by  $x_\infty = I_0$ . So we have

**Theorem 5.3.** *The genus zero part  $F_0$  of the free energy  $F$  satisfies the following differential equation:*

$$(161) \quad \frac{\partial F_0}{\partial t_0} = I_0.$$

As a corollary, we have:

**Theorem 5.4.** *The derivative  $\frac{\partial F_0}{\partial t_0}$  has the following explicit expression in  $t$ -coordinates:*

$$(162) \quad \frac{\partial F_0}{\partial t_0} = I_0 = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{p_1+\dots+p_k=k-1} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_k}}{p_k!}.$$

*Proof.* This just follows from the formula (19) for  $I_0$ .  $\square$

**Corollary 5.5.** *The derivative  $\frac{\partial F_0}{\partial t_0}$  has the following explicit formula:*

$$(163) \quad \frac{\partial F_0}{\partial t_0} = \sum_{k=1}^{\infty} \frac{1}{k(1-t_1)^k} \sum_{\substack{p_1+\dots+p_k=k-1 \\ p_j \neq 1, j=1,\dots,k}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_k}}{p_k!}.$$

*Proof.* Rewrite the polymer equation in genus zero (160) as

$$(164) \quad \frac{\partial F_0}{\partial t_0} = \frac{t_0}{1-t_1} + \sum_{n \geq 2} \frac{t_2}{1-t_1} \frac{(\frac{\partial F_0}{\partial t_0})^n}{n!}.$$

This is just (160) with  $t_i$  replaced by  $\tilde{t}_i$ , where

$$(165) \quad \tilde{t}_i = \begin{cases} \frac{t_i}{1-t_1}, & i \neq 1, \\ 0, & i = 1. \end{cases}$$

Therefore,

$$(166) \quad \frac{\partial F_0}{\partial t_0} = \sum_{k=1}^{\infty} \frac{1}{k(1-t_1)^k} \sum_{\substack{p_1+\dots+p_k=k-1 \\ p_j \neq 1, j=1,\dots,k}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_k}}{p_k!}.$$

One can also get this directly from (162) as follows:

$$\begin{aligned}
\frac{\partial F_0}{\partial t_0} &= \sum_{k=1}^{\infty} \frac{1}{k} \sum_{m=0}^{k-1} \binom{k}{m} t_1^m \sum_{\substack{p_1+\dots+p_{k-m}=k-m-1 \\ p_1, \dots, p_{k-m} \neq 1}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_{k-m}}}{p_{k-m}!} \\
&= \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \frac{1}{k+m} \binom{k+m}{m} t_1^m \sum_{\substack{p_1+\dots+p_k=k-1 \\ p_1, \dots, p_k \neq 1}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_k}}{p_k!} \\
&= \sum_{k=1}^{\infty} \frac{1}{k(1-t_1)^k} \sum_{\substack{p_1+\dots+p_k=k-1 \\ p_j \neq 1, j=1, \dots, k}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_k}}{p_k!}.
\end{aligned}$$

□

**5.7. Expressing  $F_0$  in  $I$ -coordinates.** An application of (161) is that we can have another derivation of the expression of  $F_0$  in terms of  $I$ -coordinates. Recall by (39), we have

$$\frac{\partial}{\partial t_0} = \frac{1}{1-I_1} \frac{\partial}{\partial I_0} + \sum_{l \geq 1} \frac{I_{l+1}}{1-I_1} \frac{\partial}{\partial I_l},$$

and so by (161) we have:

$$(167) \quad \frac{\partial}{\partial I_0} F_0 + \sum_{l \geq 1} I_{l+1} \frac{\partial}{\partial I_l} F_0 = (1-I_1) I_0.$$

Writing

$$(168) \quad F_0 = \sum_{j=2}^{\infty} b_j(I_1, I_2, \dots) \frac{I_0^j}{j!},$$

one gets the following recursion relations:

$$\begin{aligned}
b_2 &= (1-I_1), \\
b_j &= - \sum_{l \geq 1} I_{l+1} \frac{\partial}{\partial I_l} b_{j-1}, \quad j \geq 3.
\end{aligned}$$

We find the following solution:

$$(169) \quad F_0 = \frac{1}{2!} (1-I_1) I_0^2 + \sum_{j=3}^{\infty} (-1)^j I_{j-1} \frac{I_0^j}{j!}.$$

This is just (126) derived in §4.7.

**5.8. Proof of Theorem 3.1.** Another application of (126) is that we can now have a proof of Theorem 3.1. Recall formula (100) expresses  $F_0$  as a summation over trees:

$$F_0 = \sum_{\Gamma \text{ is a tree}} \frac{1}{|\text{Aut}(\Gamma)|} \prod_{v \in V(\Gamma)} t_{\text{val}(v)-1}.$$

Taking  $\frac{\partial}{\partial t_0}$  on both sides proves Theorem 3.1.

### 5.9. Explicit expression of $F_0$ in $t$ -coordinates.

**Theorem 5.6.** *The following formulas hold:*

$$(170) \quad F_0 = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{p_1+\dots+p_{k+1}=k-1} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_{k+1}}}{p_{k+1}!}$$

$$(171) \quad = \sum_{k=1}^{\infty} \frac{1}{k(k+1)(1-t_1)^k} \sum_{\substack{p_1+\dots+p_{k+1}=k-1 \\ p_1, \dots, p_{k+1} \neq 1}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_{k+1}}}{p_{k+1}!}.$$

*Proof.* We first rewrite (162) as follows:

$$(172) \quad \frac{\partial F_0}{\partial t_0} = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{m=1}^k \binom{k}{m} t_0^m \sum_{\substack{p_1+\dots+p_{k-m}=k-1 \\ p_1, \dots, p_{k-m} > 0}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_{k-m}}}{p_{k-m}!}.$$

Integrating once,

$$\begin{aligned} F_0 &= \sum_{k=1}^{\infty} \frac{1}{k} \sum_{m=1}^k \binom{k}{m} \frac{t_0^{m+1}}{m+1} \sum_{\substack{p_1+\dots+p_{k-m}=k-1 \\ p_1, \dots, p_{k-m} > 0}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_{k-m}}}{p_{k-m}!} \\ &= \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{m=1}^k \binom{k+1}{m+1} t_0^{m+1} \sum_{\substack{p_1+\dots+p_{k-m}=k-1 \\ p_1, \dots, p_{k-m} > 0}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_{k-m}}}{p_{k-m}!} \\ &= \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{p_1+\dots+p_{k+1}=k-1} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_{k+1}}}{p_{k+1}!}. \end{aligned}$$

Similarly, rewrite (163) as follows:

$$(173) \quad \frac{\partial F_0}{\partial t_0} = \sum_{k=1}^{\infty} \frac{1}{k(1-t_1)^k} \sum_{m=1}^k \binom{k}{m} t_0^m \sum_{\substack{p_1+\dots+p_{k-m}=k-1 \\ p_j \neq 0, 1, \quad j=1, \dots, k-m}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_k}}{p_k!}.$$

Integrating once:

$$\begin{aligned}
F_0 &= \sum_{k=1}^{\infty} \frac{1}{k(1-t_1)^k} \sum_{m=1}^k \binom{k}{m} \frac{t_0^{m+1}}{m+1} \sum_{\substack{p_1+\dots+p_{k-m}=k-1 \\ p_j \neq 0,1, \ j=1,\dots,k-m}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_k}}{p_k!} \\
&= \sum_{k=1}^{\infty} \frac{1}{k(k+1)(1-t_1)^k} \sum_{\substack{p_1+\dots+p_{k+1}=k-1 \\ p_1,\dots,p_{k+1} \neq 1}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_{k+1}}}{p_{k+1}!}.
\end{aligned}$$

□

#### 5.10. Explicit expression of higher derivatives of $F_0$ in $t$ -coordinates.

By (161) and (59) we get:

$$(174) \quad \frac{\partial^{l+1} F_0}{\partial t_0^{l+1}} = \sum_{k=1}^{\infty} (k+1) \dots (k+l-1) \sum_{\substack{p_1+\dots+p_k=k+l-1}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_k}}{p_k!}.$$

Similarly, one can differentiate

$$(175) \quad \frac{\partial F_0}{\partial t_0} = \sum_{k=1}^{\infty} \frac{1}{k(1-t_1)^k} \sum_{\substack{p_1+\dots+p_k=k-1 \\ p_j \neq 1, \ j=1,\dots,k}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_k}}{p_k!}$$

repeated as follows. First rewrite it in the following form:

$$(176) \quad \frac{\partial F_0}{\partial t_0} = \sum_{k=1}^{\infty} \frac{1}{k(1-t_1)^k} \sum_{m=1}^k \binom{k}{m} t_0^m \sum_{\substack{p_1+\dots+p_{k-m}=k-1 \\ p_j \neq 0,1, \ j=1,\dots,k-m}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_k}}{p_k!}.$$

Then one finds:

$$\begin{aligned}
\frac{\partial^2 F_0}{\partial t_0^2} &= \frac{1}{1-t_1} + \sum_{k=2}^{\infty} \frac{1}{k(1-t_1)^k} \sum_{m=1}^k \binom{k}{m} m t_0^{m-1} \sum_{\substack{p_1+\dots+p_{k-m}=k-1 \\ p_j \neq 0,1, \ j=1,\dots,k-m}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_{k-m}}}{p_{k-m}!} \\
&= \frac{1}{1-t_1} + \sum_{k=2}^{\infty} \frac{1}{(1-t_1)^k} \sum_{m=1}^k \binom{k-1}{m-1} t_0^{m-1} \sum_{\substack{p_1+\dots+p_{k-m}=k-1 \\ p_j \neq 0,1, \ j=1,\dots,k-m}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_{k-m}}}{p_{k-m}!} \\
&= \frac{1}{1-t_1} + \sum_{k=2}^{\infty} \frac{1}{(1-t_1)^k} \sum_{\substack{p_1+\dots+p_{k-1}=k-1 \\ p_j \neq 1, \ j=1,\dots,k-1}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_{k-1}}}{p_{k-1}!} \\
&= \frac{1}{1-t_1} + \sum_{k=1}^{\infty} \frac{1}{(1-t_1)^{k+1}} \sum_{\substack{p_1+\dots+p_k=k \\ p_j \neq 1, \ j=1,\dots,k}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_k}}{p_k!} \\
&= \frac{1}{1-t_1} + \sum_{k=2}^{\infty} \frac{1}{(1-t_1)^{k+1}} \sum_{\substack{p_1+\dots+p_k=k \\ p_j \neq 1, \ j=1,\dots,k}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_k}}{p_k!}.
\end{aligned}$$

One more time:

$$\begin{aligned}
\frac{\partial^3 F_0}{\partial t_0^3} &= \sum_{k=2}^{\infty} \frac{k}{(1-t_1)^{k+1}} \sum_{\substack{p_1+\dots+p_{k-1}=k \\ p_j \neq 1, \ j=1,\dots,k-1}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_{k-1}}}{p_{k-1}!} \\
&= \frac{t_2}{(1-t_1)^3} + \sum_{k=2}^{\infty} \frac{k+1}{(1-t_1)^{k+2}} \sum_{\substack{p_1+\dots+p_k=k+1 \\ p_j \neq 1, \ j=1,\dots,k}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_k}}{p_k!}.
\end{aligned}$$

Inductively one gets:

$$(177) \quad \frac{\partial^{l+1} F_0}{\partial t_0^{l+1}} = \sum_{k=1}^{\infty} \frac{(k+1) \dots (k+l-1)}{(1-t_1)^{k+l}} \sum_{\substack{p_1+\dots+p_k=k+l-1 \\ p_j \neq 1, \ j=1,\dots,k}} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_k}}{p_k!}.$$

One reason that we derive such explicit expressions for derivatives of  $F_0$  in  $t_0$  is that they appear in the Feynman rules for  $n$ -point functions derived in §9.

**5.11. Integration of the genus zero polymer equation.** Notice that  $F_0|_{t_0=0} = 0$ , so we have:

$$\begin{aligned}
& \int_0^{t_0} \left( \frac{\partial F_0}{\partial t_0} \right)^n dt_0 \\
&= t_0 \left( \frac{\partial F_0}{\partial t_0} \right)^n - n \int_0^{t_0} t_0 \left( \frac{\partial F_0}{\partial t_0} \right)^{n-1} d \frac{\partial F_0}{\partial t_0} \\
&= t_0 \left( \frac{\partial F_0}{\partial t_0} \right)^n - n \int_0^{t_0} \left( \frac{\partial F_0}{\partial t_0} - \sum_{m \geq 1} \frac{t_m}{m!} \left( \frac{\partial F_0}{\partial t_0} \right)^m \right) \left( \frac{\partial F_0}{\partial t_0} \right)^{n-1} d \frac{\partial F_0}{\partial t_0} \\
&= t_0 \left( \frac{\partial F_0}{\partial t_0} \right)^n - \frac{n}{n+1} \left( \frac{\partial F_0}{\partial t_0} \right)^{n+1} + \sum_{m \geq 1} \frac{nt_m}{m!(m+n)} \left( \frac{\partial F_0}{\partial t_0} \right)^{m+n} \\
&= t_0 I_0^n - \frac{n}{n+1} I_0^{n+1} + \sum_{m \geq 1} \frac{nt_m}{m!(m+n)} I_0^{m+n},
\end{aligned}$$

We now integrate the genus zero polymer equation (160):

$$\frac{\partial F_0}{\partial t_0} = \sum_{n \geq 0} \frac{t_n}{n!} \left( \frac{\partial F_0}{\partial t_0} \right)^n$$

to get

$$\begin{aligned}
F_0 &= \frac{t_0^2}{2} + \sum_{n=1}^{\infty} \frac{t_n}{n!} \int_0^{t_0} \left( \frac{\partial F_0}{\partial t_0} \right)^n dt_0 \\
&= \frac{t_0^2}{2} + \sum_{n=1}^{\infty} \frac{t_n}{n!} t_0 I_0^n - \sum_{n=1}^{\infty} \frac{t_n}{(n-1)!} \left( \frac{I_0^{n+1}}{n+1} - \sum_{m \geq 1} \frac{t_m}{m!} \frac{I_0^{m+n}}{m+n} \right).
\end{aligned}$$

We now show that this gives an alternative proof of the identity:

$$(178) \quad F_0 = t_0 I_0 + (t_1 - 1) \frac{I_0^2}{2!} + t_2 \frac{I_0^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{t_n - \delta_{n,1}}{(n+1)!} I_0^{n+1}.$$

It suffices to show that

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{t_n}{(n-1)!} \sum_{m \geq 1} \frac{t_m}{m!} \frac{I_0^{m+n}}{m+n} \\
&= \sum_{n=0}^{\infty} \frac{t_n - \delta_{n,1}}{(n+1)!} I_0^{n+1} + \sum_{n=1}^{\infty} \frac{t_n}{(n-1)!} \frac{I_0^{n+1}}{n+1} - \frac{t_0^2}{2} - \sum_{n=1}^{\infty} \frac{t_n}{n!} t_0 I_0^n.
\end{aligned}$$



The right-hand side can be rewritten as follows:

$$\begin{aligned} & \sum_{n=0}^{\infty} t_n \frac{I_0^{n+1}}{(n+1)!} - \frac{I_0^2}{2} + \sum_{n=1}^{\infty} n t_n \frac{I_0^{n+1}}{(n+1)!} - \frac{t_0^2}{2} - t_0(I_0 - t_0) \\ &= \frac{1}{2}(I_0 - t_0)^2; \end{aligned}$$

the left-hand side can be dealt with as follows:

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{t_n}{(n-1)!} \sum_{m \geq 1} \frac{t_m}{m!} \frac{I_0^{m+n}}{m+n} = \sum_{m,n=1}^{\infty} \frac{t_n}{(n-1)!} \frac{t_m}{m!} \frac{I_0^{m+n}}{m+n} \\ &= \frac{1}{2} \sum_{m,n=1}^{\infty} \left( \frac{t_n}{(n-1)!} \frac{t_m}{m!} + \frac{t_m}{(m-1)!} \frac{t_n}{n!} \right) \frac{I_0^{m+n}}{m+n} \\ &= \frac{1}{2} \sum_{m,n=1}^{\infty} \frac{t_n}{n!} \frac{t_m}{m!} I_0^{m+n} = \frac{1}{2} \left( \sum_{n \geq 1} \frac{t_n}{n!} I_0^n \right)^2 = \frac{1}{2}(I_0 - t_0)^2. \end{aligned}$$

This finishes the proof.

## 6. VIRASORO CONSTRAINTS FOR TOPOLOGICAL 1D GRAVITY

In this Section we study the applications of Virasoro constraints for topological 1D gravity derived in [8]. These include another derivation of the formulas for  $F_g$  in  $I$ -coordinates.

**6.1. Virasoro constraints from flow equation and polymer equation.** By the polymer equation

$$\sum_{n \geq 0} \frac{t_n - \delta_{n,1}}{n!} \lambda^{2n} \frac{\partial^n}{\partial t_0^n} Z = 0.$$

and the flow equation:

$$\frac{\partial Z}{\partial t_n} = \frac{\lambda^{2n}}{(n+1)!} \frac{\partial^{n+1} Z}{\partial t_0^{n+1}}$$

one gets the *puncture equation* for topological 1D gravity:

$$(179) \quad (t_0 + \sum_{n \geq 1} (t_n - \delta_{n,1}) \lambda^2 \frac{\partial}{\partial t_{n-1}}) Z = 0.$$

Take derivative in  $t_0$  and apply the flow equation again:

$$(180) \quad \left( 1 + \sum_{n \geq 0} (n+1)(t_n - \delta_{n,1}) \frac{\partial}{\partial t_n} \right) Z = 0.$$

This is the *dilaton equation* for topological 1D gravity. Take the  $m+1$ -th derivative in  $t_0$  by Leibnitz formula:

$$\left( (m+1) \frac{\partial^m}{\partial t_0^m} + \sum_{n \geq 0} \frac{t_n - \delta_{n,1}}{n!} \lambda^{2n} \frac{\partial^{m+n+1}}{\partial t_0^{m+n+1}} \right) Z = 0,$$

and apply the flow equation:

$$\begin{aligned} & \left( (m+1)! \lambda^{-2(m-1)} \frac{\partial}{\partial t_{m-1}} \right. \\ & \left. + \sum_{n \geq 0} \frac{(m+n+1)! (t_n - \delta_{n,1})}{n!} \lambda^{2n-2(m+n)} \frac{\partial}{\partial t_{m+n}} \right) Z = 0. \end{aligned}$$

This is how the authors of [8] derived the following:

**Theorem 6.1.** *The partition function  $Z$  of topological 1D gravity satisfies the following equations for  $m \geq -1$ :*

$$(181) \quad L_m Z = 0,$$

where

$$(182) \quad L_{-1} = \frac{t_0}{\lambda^2} + \sum_{m \geq 1} (t_m - \delta_{m,1}) \frac{\partial}{\partial t_{m-1}},$$

$$(183) \quad L_0 = 1 + \sum_{m \geq 0} (t_m - \delta_{m,1}) (m+1) \frac{\partial}{\partial t_m},$$

$$(184) \quad L_m = \lambda^2 (m+1)! \frac{\partial}{\partial t_{m-1}} + \sum_{n \geq 1} (t_{n-1} - \delta_{n,2}) \frac{(m+n)!}{(n-1)!} \frac{\partial}{\partial t_{m+n-1}},$$

for  $m \geq 1$ .

**Theorem 6.2.** *The operators  $\{L_n\}_{n \geq -1}$  satisfies the Virasoro commutation relations:*

$$(185) \quad [L_m, L_n] = (m-n) L_{m+n}.$$

*Proof.* This can be established by a straightforward calculation. Another proof given in [8] will be discussed in next Section.  $\square$

**6.2. An application of the dilaton equation.** The dilaton equation can be rewritten as

$$(186) \quad \frac{\partial F}{\partial t_1} = \sum_{m \geq 0} \frac{m+1}{2} t_m \frac{\partial F}{\partial t_m} + \frac{1}{2}.$$

In terms of correlators,

$$(187) \quad \langle \tau_1 \rangle_1 = \frac{1}{2},$$

$$(188) \quad \langle \tau_1 \prod_{j=1}^n \tau_{a_j} \rangle_g = \sum_{j=1}^n \frac{a_j + 1}{2} \langle \prod_{j=1}^n \tau_{a_j} \rangle_g.$$

Therefore,

$$(189) \quad \langle \tau_1^m \rangle_1 = \frac{1}{2}(m-1)!,$$

and for  $a_2, \dots, a_n \neq 1$  which satisfies the selection rule (119),

$$a_1 + \dots + a_n = 2g - 2 + n,$$

we have

$$(190) \quad \langle \tau_m \prod_{j=1}^n \tau_{a_j} \rangle_g = \prod_{k=0}^{m-1} (g - 1 + n + k) \cdot \langle \prod_{j=1}^n \tau_{a_j} \rangle_g.$$

It follows that we have

**Theorem 6.3.** *The free energy of topological 1D gravity can be rewritten in the following form:*

$$(191) \quad F = \frac{1}{2} \log(1 - t_1) + \sum_{\substack{g, n \geq 0 \\ 2g-2+n > 0}} \sum_{\substack{a_2, \dots, a_n \neq 1 \\ a_1 + \dots + a_n = 2g-2+n}} \frac{\langle \prod_{j=1}^n \tau_{a_j} \rangle_g}{(1 - t_1)^{g-1+n}}.$$

**6.3. The operator  $L_{-1}$  in I-coordinates.** By (34) and (28), we have

$$(192) \quad L_{-1} = -\frac{\partial}{\partial I_0} + \frac{1}{\lambda^2} \sum_{n=0}^{\infty} \frac{(-1)^n I_0^n}{n!} I_n.$$

This can also be checked by (39):

$$\begin{aligned}
\sum_{k \geq 0} t_{k+1} \frac{\partial}{\partial t_k} &= \sum_{k \geq 0} t_{k+1} \left( \frac{1}{1-I_1} \frac{I_0^k}{k!} \frac{\partial}{\partial I_0} + \frac{I_0^k}{k!} \sum_{l \geq 1} \frac{I_{l+1}}{1-I_1} \frac{\partial}{\partial I_l} + \sum_{1 \leq l \leq k} \frac{I_0^{k-l}}{(k-l)!} \frac{\partial}{\partial I_l} \right) \\
&= \frac{1}{1-I_1} \sum_{k \geq 0} t_{k+1} \frac{I_0^k}{k!} \cdot \frac{\partial}{\partial I_0} + \sum_{k \geq 0} t_{k+1} \frac{I_0^k}{k!} \sum_{l \geq 1} \frac{I_{l+1}}{1-I_1} \frac{\partial}{\partial I_l} \\
&\quad + \sum_{k \geq 0} t_{k+1} \sum_{1 \leq l \leq k} \frac{I_0^{k-l}}{(k-l)!} \frac{\partial}{\partial I_l} \\
&= \frac{I_1}{1-I_1} \cdot \frac{\partial}{\partial I_0} + \sum_{l \geq 1} \frac{I_1 I_{l+1}}{1-I_1} \frac{\partial}{\partial I_l} + \sum_{l=1}^{\infty} \sum_{k \geq l} t_{k+1} \frac{I_0^{k-l}}{(k-l)!} \frac{\partial}{\partial I_l} \\
&= \frac{I_1}{1-I_1} \cdot \frac{\partial}{\partial I_0} + \sum_{l \geq 1} \frac{I_1 I_{l+1}}{1-I_1} \frac{\partial}{\partial I_l} + \sum_{l=1}^{\infty} I_{l+1} \frac{\partial}{\partial I_l} \\
&= \frac{I_1}{1-I_1} \cdot \frac{\partial}{\partial I_0} + \sum_{l \geq 1} \frac{I_{l+1}}{1-I_1} \frac{\partial}{\partial I_l}.
\end{aligned}$$

It follows from (192) that

$$(193) \quad \frac{\partial F_0}{\partial I_0} = \sum_{n=0}^{\infty} \frac{(-1)^n I_0^n}{n!} I_n,$$

$$(194) \quad \frac{\partial F_g}{\partial I_0} = 0, \quad g \geq 1.$$

Therefore, we rederive the following result obtained in §4.7.

**Theorem 6.4.** *The genus zero part of the free energy  $F_0$  of topological 1D gravity is given in  $I$ -coordinates by:*

$$(195) \quad F_0 = \frac{1}{2} I_0^2 + \sum_{n=0}^{\infty} \frac{(-1)^n I_0^{n+1}}{(n+1)!} I_n.$$

When  $g \geq 1$ ,  $F_g$  is independent of  $I_0$ .

#### 6.4. Dilaton operator in $I$ -coordinates.

**Lemma 6.5.** *The dilaton operator  $L_0$  is given in  $I$ -coordinates by:*

$$(196) \quad L_0 = -I_0 \frac{\partial}{\partial I_0} - 2 \frac{\partial}{\partial I_1} + \sum_{l \geq 1} (l+1) I_l \frac{\partial}{\partial I_l} + 1.$$

*Proof.* By (39),

$$\begin{aligned}
& \sum_{k \geq 0} (k+1)t_k \frac{\partial}{\partial t_k} = \sum_{k, l \geq 0} (k+1)t_k \left( \frac{1}{1-I_1} \frac{I_0^k}{k!} \frac{\partial}{\partial I_0} \right. \\
& + \left. \frac{I_0^k}{k!} \sum_{l \geq 1} \frac{I_{l+1}}{1-I_1} \frac{\partial}{\partial I_l} + \sum_{1 \leq l \leq k} \frac{I_0^{k-l}}{(k-l)!} \frac{\partial}{\partial I_l} \right) \\
& = \frac{I_0 I_1 + I_0}{1-I_1} \frac{\partial}{\partial I_0} + \sum_{l \geq 1} \left( \frac{(I_0 I_1 + I_0) I_{l+1}}{1-I_1} + (I_0 I_{l+1} + (l+1) I_l) \right) \frac{\partial}{\partial I_l} \\
& = \frac{I_0 I_1 + I_0}{1-I_1} \frac{\partial}{\partial I_0} + \sum_{l \geq 1} \left( \frac{2 I_0 I_{l+1}}{1-I_1} + (l+1) I_l \right) \frac{\partial}{\partial I_l}.
\end{aligned}$$

And by (39) for  $k = 1$ ,

$$\frac{\partial}{\partial t_1} = \frac{I_0}{1-I_1} \frac{\partial}{\partial I_0} + \left( \frac{I_2 I_0}{1-I_1} + 1 \right) \frac{\partial}{\partial I_1} + \sum_{l \geq 2} \frac{I_{l+1} I_0}{1-I_1} \frac{\partial}{\partial I_l}.$$

Therefore,

$$\begin{aligned}
L_0 &= -2 \frac{\partial}{\partial t_1} + \sum_{k \geq 0} (k+1)t_k \frac{\partial}{\partial t_k} + 1 \\
&= -2 \left( \frac{I_0}{1-I_1} \frac{\partial}{\partial I_0} + \left( \frac{I_2 I_0}{1-I_1} + 1 \right) \frac{\partial}{\partial I_1} + \sum_{l \geq 2} \frac{I_{l+1} I_0}{1-I_1} \frac{\partial}{\partial I_l} \right) \\
&+ \frac{I_0 I_1 + I_0}{1-I_1} \frac{\partial}{\partial I_0} + \sum_{l \geq 1} \left( \frac{2 I_0 I_{l+1}}{1-I_1} + (l+1) I_l \right) \frac{\partial}{\partial I_l} \\
&= -I_0 \frac{\partial}{\partial I_0} - 2 \frac{\partial}{\partial I_1} + \sum_{l \geq 1} (l+1) I_l \frac{\partial}{\partial I_l} + 1.
\end{aligned}$$

□

From the dilaton equation, one gets

$$(197) \quad \frac{\partial F_0}{\partial I_1} = \sum_{l \geq 1} \frac{l+1}{2} I_l \frac{\partial F_0}{\partial I_l} + \frac{1}{2} \frac{\partial F_0}{\partial I_0},$$

$$(198) \quad \frac{\partial F_1}{\partial I_1} = \sum_{l \geq 1} \frac{l+1}{2} I_l \frac{\partial F_1}{\partial I_l} + \frac{1}{2},$$

$$(199) \quad \frac{\partial F_g}{\partial I_1} = \sum_{l \geq 1} \frac{l+1}{2} I_l \frac{\partial F_g}{\partial I_l}, \quad g \geq 2.$$

**6.5. Solution in positive genera.** In this Subsection we will solve the dilaton equation for  $g \geq 1$ . We rederive the following result obtained in §4.7.

**Theorem 6.6.** *In  $I$ -coordinates we have*

$$(200) \quad F_1 = \frac{1}{2} \ln \frac{1}{1 - I_1}$$

and for  $g \geq 1$ ,

$$(201) \quad F_g = \sum_{\sum_{j=2}^{2g-1} \frac{j-1}{2} l_j = g-1} \langle \tau_2^{l_2} \cdots \tau_{2g-1}^{l_{2g-1}} \rangle_g \prod_{j=2}^{2g-1} \frac{1}{l_j!} \left( \frac{I_j}{(1 - I_1)^{(j+1)/2}} \right)^{l_j}.$$

*Proof.* By the puncture equation we have already seen that  $F_g$  does not depend on  $I_0$  when  $g \geq 1$ . Write  $F_g$  as formal power series in  $I_1$ , with coefficients a priori formal series in  $I_2, I_3, \dots$ ,

$$(202) \quad F_g = a_{0,g}(I_2, I_3, \dots) + a_{1,g}(I_2, I_3, \dots)I_1 + \cdots.$$

Write

$$(203) \quad a_{0,g} = \sum \alpha_{l_2, \dots, l_m} \frac{I_2^{l_2}}{l_2!} \cdots \frac{I_m^{l_m}}{l_m!}$$

By comparing the coefficients of  $\frac{I_2^{l_2}}{l_2!} \cdots \frac{I_m^{l_m}}{l_m!}$  on both sides of (202), it is easy to see that:

$$\alpha_{l_2, \dots, l_m} = \frac{\partial^{l_2 + \cdots + l_m} F_g}{(\partial t_2)^{l_2} \cdots (\partial t_m)^{l_m}} \Big|_{\mathbf{t}=0} = \langle \tau_2^{l_2} \cdots \tau_m^{l_m} \rangle_g.$$

This vanishes unless the following selection rule is satisfied:

$$(204) \quad \sum_{j=2}^m j l_j = 2g - 2 + \sum_{j=2}^m l_j.$$

Assume  $l_m \geq 1$ , then

$$m - 1 \leq \sum_{j=2}^m (j - 1) l_j = 2g - 2,$$

hence

$$m \leq 2g - 1.$$

Therefore,

$$(205) \quad a_{0,g} = \sum \langle \tau_2^{l_2} \cdots \tau_{2g-1}^{l_{2g-1}} \rangle_g \frac{I_2^{l_2}}{l_2!} \cdots \frac{I_{2g-1}^{l_{2g-1}}}{l_{2g-1}!},$$

where the summation is taken over all nonnegative integers  $l_2, \dots, l_{2g-1}$  such that

$$(206) \quad \sum_{j=2}^{2g-1} \frac{j+1}{2} l_j = g-1 + \sum_{j=2}^{2g-1} l_j.$$

Let

$$\tilde{E} = \sum_{l \geq 2} \frac{l+1}{2} I_l \frac{\partial}{\partial I_l}.$$

The the equation (198) gives us the following recursion relations:

$$(207) \quad a_{1,g} = \tilde{E} a_{0,g} + \frac{\delta_{g,1}}{2},$$

$$(208) \quad n a_{n,g} = (n-1) a_{n-1,g} + \tilde{E} a_{n-1,g}, \quad n \geq 2.$$

When  $g = 1$ ,  $a_{0,1} = 0$ . One easily sees that  $a_{n,1} = \frac{1}{2n}$ . Therefore,

$$F_1 = \sum_{n \geq 1} \frac{1}{2n} I_1^n = \frac{1}{2} \ln \frac{1}{1 - I_1}.$$

When  $g > 1$ , one finds

$$a_{n,g} = \sum_{\sum_{j=2}^{2g-1} \frac{j-1}{2} l_j = g-1} \langle \tau_2^{l_2} \cdots \tau_{2g-1}^{l_{2g-1}} \rangle_g (-1)^n \binom{-(g-1 + \sum_{j=2}^{2g-1} l_j)}{n} \frac{I_2^{l_2}}{l_2!} \cdots \frac{I_{2g-2}^{l_{2g-2}}}{l_{2g-2}!}.$$

This proves:

$$\begin{aligned} F_g &= \sum_{\sum_{j=2}^{2g-1} \frac{j-1}{2} l_j = g-1} \langle \tau_2^{l_2} \cdots \tau_{2g-1}^{l_{2g-1}} \rangle_g \frac{1}{(1 - I_1)^{g-1 + \sum_{j=2}^{2g-1} l_j}} \prod_{j=2}^{2g-1} \frac{I_j^{l_j}}{l_j!} \\ &= \sum_{\sum_{j=2}^{2g-1} \frac{j-1}{2} l_j = g-1} \langle \tau_2^{l_2} \cdots \tau_{2g-1}^{l_{2g-1}} \rangle_g \prod_{j=2}^{2g-1} \frac{1}{l_j!} \left( \frac{I_j}{(1 - I_1)^{(j+1)/2}} \right)^{l_j}. \end{aligned}$$

□

For example,

$$\begin{aligned}
F_2 &= \frac{1}{2} \langle \tau_2^2 \rangle_2 \frac{I_2^2}{(1-I_1)^3} + \langle \tau_3 \rangle_2 \frac{I_3}{(1-I_1)^2}, \\
F_3 &= \langle \tau_2^4 \rangle_3 \cdot \frac{1}{4!} \frac{I_2^4}{(1-I_1)^6} + \langle \tau_2^2 \tau_3 \rangle_3 \cdot \frac{1}{2!} \frac{I_2^2 I_3}{(1-I_1)^5} + \langle \tau_3^2 \rangle_3 \cdot \frac{1}{2!} \frac{I_3^2}{(1-I_1)^4} \\
&\quad + \langle \tau_2 \tau_4 \rangle_3 \cdot \frac{I_2 I_4}{(1-I_1)^4} + \langle \tau_5 \rangle_3 \frac{I_5}{(1-I_1)^3}, \\
F_4 &= \langle \tau_2^6 \rangle_4 \cdot \frac{1}{6!} \frac{I_2^6}{(1-I_1)^9} + \langle \tau_2^4 \tau_3 \rangle_4 \cdot \frac{1}{4!} \frac{I_2^4 I_3}{(1-I_1)^8} + \langle \tau_2^2 \tau_3^2 \rangle_4 \cdot \frac{1}{2!2!} \frac{I_2^2 I_3^2}{(1-I_1)^7} \\
&\quad + \langle \tau_3^3 \rangle_4 \frac{1}{3!} \frac{I_3^3}{(1-I_1)^6} + \langle \tau_2^3 \tau_4 \rangle_3 \cdot \frac{1}{3!} \frac{I_2^3 I_4}{(1-I_1)^7} + \langle \tau_2 \tau_3 \tau_4 \rangle_4 \cdot \frac{I_2 I_3 I_4}{(1-I_1)^6} \\
&\quad + \langle \tau_4^2 \rangle_4 \cdot \frac{1}{2!} \frac{I_4^2}{(1-I_1)^5} + \langle \tau_2^2 \tau_5 \rangle_4 \cdot \frac{1}{2!} \frac{I_2^2 I_5}{(1-I_1)^6} + \langle \tau_2 \tau_6 \rangle_4 \cdot \frac{I_2 I_6}{(1-I_1)^5} \\
&\quad + \langle \tau_7 \rangle_4 \cdot \frac{I_7}{(1-I_1)^4}.
\end{aligned}$$

For the relevant correlators, see §4.5.

## 7. OPERATOR ALGEBRA OF TOPOLOGICAL 1D GRAVITY AND W-CONSTRAINTS

In this Section we study the operator algebra that leads to the Virasoro constraints and W-constraints of topological 1D gravity. We also present a different version of Virasoro constraints.

**7.1. Virasoro constraints as Dyson-Schwinger equations.** For  $n \geq -1$ , by (90) one has:

$$(209) \quad \frac{1}{\sqrt{2\pi\lambda}} \int dx \cdot \frac{\partial}{\partial x} \left( \frac{x^{n+1}}{(n+1)!} \cdot \exp \frac{1}{\lambda^2} \left( -\frac{1}{2}x^2 + \sum_{n \geq 1} t_{n-1} \frac{x^n}{n!} \right) \right) = 0.$$

Rewrite the left-hand side as follows: For  $n \geq 1$ ,

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi\lambda}} \int dx \cdot \left( \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)! \lambda^2} \cdot \sum_{m \geq 1} (t_{m-1} - \delta_{m,2}) \frac{x^{m-1}}{(m-1)!} \right) \\
&\cdot \exp \frac{1}{\lambda^2} \left( -\frac{1}{2}x^2 + \sum_{k \geq 1} t_{n-1} \frac{x^k}{k!} \right) \\
&= \left( \lambda^2 \frac{\partial}{\partial t_{n-1}} + \sum_{m \geq 1} (t_{m-1} - \delta_{m,2}) \frac{(m+n)!}{(n+1)!(m-1)!} \frac{\partial}{\partial t_{m+n-1}} \right) Z;
\end{aligned}$$



there are two exceptional cases: for  $n = -1$ ,

$$\begin{aligned} & \frac{1}{\sqrt{2\pi\lambda}} \int dx \cdot \left( \frac{1}{\lambda^2} \cdot \sum_{m \geq 1} (t_{m-1} - \delta_{m,2}) \frac{x^{m-1}}{(m-1)!} \right) \\ & \cdot \exp \frac{1}{\lambda^2} \left( -\frac{1}{2}x^2 + \sum_{k \geq 1} t_{k-1} \frac{x^k}{k!} \right) \\ & = \left( \frac{t_0}{\lambda^2} + \sum_{m \geq 1} (t_m - \delta_{m,1}) \frac{\partial}{\partial t_{m-1}} \right) Z; \end{aligned}$$

and for  $n = 0$ ,

$$\begin{aligned} & \frac{1}{\sqrt{2\pi\lambda}} \int dx \cdot \left( 1 + \frac{x}{\lambda^2} \cdot \sum_{m \geq 1} (t_{m-1} - \delta_{m,2}) \frac{x^{m-1}}{(m-1)!} \right) \\ & \cdot \exp \frac{1}{\lambda^2} \left( -\frac{1}{2}x^2 + \sum_{k \geq 1} t_{k-1} \frac{x^k}{k!} \right) \\ & = \left( 1 + \sum_{m \geq 0} (t_m - \delta_{m,1})(m+1) \frac{\partial}{\partial t_m} \right) Z. \end{aligned}$$

**7.2. Virasoro constraints from loop equation.** By (90) one has:

$$(210) \quad \frac{1}{\sqrt{2\pi\lambda}} \int dx \cdot \frac{\partial}{\partial x} \left( \frac{1}{z-x} \cdot \exp \frac{1}{\lambda^2} \left( -\frac{1}{2}x^2 + \sum_{n \geq 1} t_{n-1} \frac{x^n}{n!} \right) \right) = 0.$$

This is called the *loop equation* of the topological 1D gravity. By the expansion

$$(211) \quad \frac{1}{z-x} = \sum_{n=0}^{\infty} \frac{x^n}{z^{n+1}},$$

one can rederive the Virasoro constraints by consider the coefficients of  $\frac{1}{z^{n+2}}$  for  $n \geq -1$ .

**7.3. W-constraints.** By (90) one has:

$$(212) \quad \frac{1}{\sqrt{2\pi\lambda}} \int dx \cdot \frac{\partial^k}{\partial x^k} \left( \frac{x^{n+1}}{(n+1)!} \cdot \exp \frac{1}{\lambda^2} \left( -\frac{1}{2}x^2 + \sum_{n \geq 1} t_{n-1} \frac{x^n}{n!} \right) \right) = 0.$$

Use Leibniz formula to rewrite the left-hand side as follows:

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi}\lambda} \int dx \cdot \left( \sum_{j=0}^k \binom{k}{j} \frac{\partial^{k-j}}{\partial x^{k-j}} \left( \frac{x^{n+1}}{(n+1)!} \right) \right. \\
& \quad \cdot \frac{\partial^j}{\partial x^j} \exp \frac{1}{\lambda^2} \left( -\frac{1}{2}x^2 + \sum_{n \geq 1} t_{n-1} \frac{x^n}{n!} \right) \Bigg) \\
&= \frac{1}{\sqrt{2\pi}\lambda} \int dx \cdot \left( \sum_{j=0}^k \binom{k}{j} (k-j)! \binom{n+1-k+j}{k-j} \frac{x^{n+1-k+j}}{(n+1)!} \right. \\
& \quad \cdot \frac{\partial^j}{\partial x^j} \exp \frac{1}{\lambda^2} \left( -\frac{1}{2}x^2 + \sum_{n \geq 1} t_{n-1} \frac{x^n}{n!} \right) \Bigg) \\
&= \frac{1}{\sqrt{2\pi}\lambda} \int dx \cdot \left( \sum_{j=0}^k \binom{k}{j} (k-j)! \binom{n+1-k+j}{k-j} \frac{x^{n+1-k+j}}{(n+1)!} \right. \\
& \quad \cdot \sum_{l \geq 0} A_{j,l}(\mathbf{t}) \frac{x^l}{l!} \cdot \exp \frac{1}{\lambda^2} \left( -\frac{1}{2}x^2 + \sum_{n \geq 1} t_{n-1} \frac{x^n}{n!} \right) \Bigg),
\end{aligned}$$

where

$$\begin{aligned}
& \frac{\partial^j}{\partial x^j} \exp \frac{1}{\lambda^2} \left( -\frac{1}{2}x^2 + \sum_{n \geq 1} t_{n-1} \frac{x^n}{n!} \right) \\
&= \sum_{l \geq 0} A_{j,l}(\mathbf{t}) \frac{x^l}{l!} \cdot \exp \frac{1}{\lambda^2} \left( -\frac{1}{2}x^2 + \sum_{n \geq 1} t_{n-1} \frac{x^n}{n!} \right).
\end{aligned}$$

Therefore, one gets a constraint:

$$\begin{aligned}
(213) \quad & \sum_{j=0}^k \binom{k}{j} (k-j)! \binom{n+1-k+j}{k-j} \frac{x^{n+1-k+j}}{(n+1)!} \\
& \cdot \sum_{l \geq 0} A_{j,l}(\mathbf{t}) \frac{x^l}{l!} (n+1-k+j+l)! \frac{\partial}{\partial t_{n-k+j+l}} Z = 0.
\end{aligned}$$

**7.4. Operator algebra.** Denote by  $\mathcal{A}$  the algebra of differential operators with polynomial coefficients:

$$(214) \quad \sum_{m,n \geq 0} a_{m,n} x^m \partial_x^n$$

where  $a_{m,n} = 0$  for  $m \gg 0$  or  $n \gg 0$ . One can take either  $\{x^m \partial_x^n\}_{m,n \geq 0}$  or  $\{\partial_x^n x^m\}_{m,n \geq 0}$  as a basis. By the Leibniz formula, they are related as

follows:

$$(215) \quad \partial_x^n x^m = \sum_{j=0}^n j! \binom{n}{j} \binom{m}{j} x^{m-j} \partial_x^{n-j}.$$

By induction one can also show that:

$$(216) \quad x^m \partial_x^n = \sum_{j=0}^{n-1} (-1)^j j! \binom{n-1}{j} \binom{m}{j} \partial_x^{n-j} x^{m-j}.$$

With these formulas, one gets the structure constants of the operator algebra  $\mathcal{A}$ :

$$(217) \quad x^{m_1} \partial_x^{n_1} \cdot x^{m_2} \partial_x^{n_2} = \sum_{j=0}^{n_1} j! \binom{n_1}{j} \binom{m_2}{j} x^{m_1+m_2-j} \partial_x^{n_1+n_2-j},$$

and

$$(218) \quad \partial_x^{n_1} x^{m_1} \cdot \partial_x^{n_2} x^{m_2} = \sum_{j=0}^{n_2-1} (-1)^j \binom{n_2-1}{j} \binom{m_1}{j} \partial_x^{n_1+n_2-j} x^{m_1+m_2-j}.$$

The operator algebra  $\mathcal{A}$  is associative, but not commutative. The commutators of elements in  $\mathcal{A}$  are given by the above formula for structure constants:

$$(219) \quad [x^{m_1} \partial_x^{n_1}, x^{m_2} \partial_x^{n_2}] = \sum_{j \geq 1} j! \left( \binom{n_1}{j} \binom{m_2}{j} - \binom{n_2}{j} \binom{m_1}{j} \right) x^{m_1+m_2-j} \partial_x^{n_1+n_2-j},$$

and

$$(220) \quad \begin{aligned} & [\partial_x^{n_1} x^{m_1}, \partial_x^{n_2} x^{m_2}] \\ &= \sum_{j \geq 1} (-1)^j \left( \binom{n_2-1}{j} \binom{m_1}{j} - \binom{n_1-1}{j} \binom{m_2}{j} \right) \partial_x^{n_1+n_2-j} x^{m_1+m_2-j}. \end{aligned}$$

In particular, when  $n_1 = n_2 = 1$ ,

$$(221) \quad [\partial_x x^{m_1}, \partial_x x^{m_2}] = (m_2 - m_1) \partial_x x^{m_1+m_2-1}.$$

**7.5. Representation of  $\mathcal{A}$ .** Let us consider the following natural representation of the operator algebra on the following space:

$$(222) \quad \mathcal{V} = \left\{ \sum_{j=0}^{\infty} a_j(\mathbf{t}; \lambda) x^j \cdot |0\rangle : a_j \in \mathbb{C}[[\mathbf{t}; \lambda^{-2}, \lambda^2]] \right\}.$$

where

$$(223) \quad |0\rangle = \exp \frac{1}{\lambda^2} \left( -\frac{1}{2}x^2 + \sum_{n \geq 1} t_{n-1} \frac{x^n}{n!} \right)$$

is understood as the vacuum. On  $\mathcal{V}$ , one also has the actions by the operators  $\frac{\partial}{\partial t_n}$  and multiplications by  $t_n$ , so  $\mathcal{V}$  actually admits a representation of  $\mathcal{A} \otimes \mathcal{B}$ , where  $\mathcal{B}$  is the space of differential operators in  $t_0, t_1, \dots, t_n, \dots$ , i.e., operators of the form

$$(224) \quad \sum \alpha_{i_1, \dots, i_n} \frac{\partial}{\partial t_{i_1}} \cdots \frac{\partial}{\partial t_{i_n}},$$

where the coefficients  $\alpha_{i_1, \dots, i_n} \in \mathbb{C}[t_0, t_1, \dots, \lambda^2, \lambda^{-2}]$ . With the actions of  $\mathcal{B}$  one sees that the vacuum is not an uninteresting object that contains nothing, but instead it contains everything in the sense that one can get every vector in  $\mathcal{V}$  by applying an operator in  $\mathcal{B}$  on  $|0\rangle$ .

For convenience of notations, we set

$$(225) \quad \frac{\partial}{\partial t_{-1}} = \frac{1}{\lambda^2} \cdot$$

**7.6. The action of  $\mathcal{A}$  on  $|0\rangle$ .** Let us now examine the action of operators in  $\mathcal{A}$  on  $|0\rangle$ . We will need the following notations introduced by Itzykson-Zuber [?] in their study of 2D topological gravity:

$$(226) \quad J_k = \sum_{n \geq 0} t_{n+k} \frac{x^n}{n!}.$$

Define

$$(227) \quad \tilde{J}_k = I_k - \delta_{k,0}x - \delta_{k,1}.$$

Let

$$(228) \quad S = -\frac{x^2}{2} + \sum_{n \geq 1} t_{n-1} \frac{x^n}{n!}.$$

Then

$$(229) \quad \tilde{J}_k = \frac{\partial^k}{\partial x^k} S.$$

In particular,

$$(230) \quad \tilde{J}_{k+1} = \partial_x \tilde{J}_k.$$

By induction one can easily get the following:

**Lemma 7.1.** *For  $n \geq 0$ ,*

$$(231) \quad \partial_x^n |0\rangle = D_n(\tilde{I}_0, \dots, \tilde{I}_{n-1})|0\rangle,$$

*where  $D_n$  is a polynomial in  $\tilde{J}_0, \dots, \tilde{J}_{n-1}$  recursively defined as follows:*

$$(232) \quad D_0 = 1,$$

$$(233) \quad D_{n+1} = PD_n, \quad n \neq 0,$$

*where*

$$(234) \quad P = \sum_{j \geq 0} \tilde{J}_{j+1} \frac{\partial}{\partial \tilde{J}_j} + \frac{\tilde{J}_0}{\lambda^2}.$$

*Remark 7.2.* If we define

$$(235) \quad \frac{\partial}{\partial \tilde{J}_{-1}} = \frac{1}{\lambda^2},$$

then we can write  $P$  more compactly as

$$(236) \quad P = \sum_{j \geq 0} \tilde{J}_j \frac{\partial}{\partial \tilde{J}_{j-1}}.$$

For example,

$$\begin{aligned} D_1 &= \frac{1}{\lambda^2} \tilde{J}_0, \\ D_2 &= \frac{1}{\lambda^2} \tilde{J}_1 + \frac{1}{\lambda^4} \tilde{J}_0^2, \\ D_3 &= \frac{1}{\lambda^2} \tilde{J}_2 + \frac{3}{\lambda^4} \tilde{J}_0 \tilde{J}_1 + \frac{1}{\lambda^6} \tilde{J}_0^3, \\ D_4 &= \frac{1}{\lambda^2} \tilde{J}_3 + \frac{4}{\lambda^4} \tilde{J}_0 \tilde{J}_2 + \frac{3}{\lambda^4} \tilde{J}_1^2 + \frac{6}{\lambda^6} \tilde{J}_0^2 \tilde{J}_1 + \frac{1}{\lambda^8} \tilde{J}_0^4. \end{aligned}$$

The recursive procedure of finding  $D_n$  is very similar to the procedure of finding the decomposition of  $V^{\otimes n}$  into irreducible representations by the Littlewood-Richardson rule. Indeed, if one let  $p_n = \frac{1}{\lambda^2} \tilde{J}_{n-1}$ , then each  $D_n$  is a polynomial in  $p_1, \dots, p_n$ :

$$(237) \quad D_n = \sum_{|\mu|=n} a_\mu p_\mu,$$

where the summation is over all Young diagrams with  $n$  boxes. In other words, we associate a partition of  $n$  to each monomial in  $D_n$ , and represent the partition by its Young diagram. Then we have

$$(238) \quad Pp_\mu = \sum_{|\nu|=n+1} \alpha_\nu p_\nu,$$

the right-hand side of which can be obtained as follows: Suppose that the Young diagram of  $\mu$  has  $l$  rows, write down all the  $l + 1$  possible ways to either add a box from the right on a row, or add a new row with one box from the bottom. It is possible that not all the result diagrams are Young diagrams, when they are not, simply switch the rows to make it a Young diagram, this gives rise to the coefficients  $\alpha_\nu$ . For example,

$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \times \begin{array}{|c|} \hline \square \\ \hline \end{array} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \\ \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} = 2 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} + \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}.$$

**Theorem 7.3.** For  $n \geq 1$ ,  $D_n$  is given by the following explicit formula:

$$(239) \quad D_n = n! \sum_{\substack{m_1, \dots, m_n \geq 0 \\ \sum_{j=1}^n m_j = n}} \prod_{j=1}^n \frac{\tilde{I}_{j-1}^{m_j}}{(j!)^{m_j} m_j!}.$$

*Proof.* One can directly check that the right-hand side satisfies the recursion relations (233) and the initial value (232).  $\square$

**7.7. Converting to actions of operators in  $\mathcal{B}$  on  $|0\rangle$ .** We begin with the action of  $x^m$  on  $|0\rangle$ . From the definition of  $|0\rangle$ , it is easy to see that

**Lemma 7.4.** For  $m \geq 0$ ,

$$(240) \quad \frac{x^m}{m!} |0\rangle = \lambda^2 \frac{\partial}{\partial t_{m-1}} |0\rangle.$$

**Corollary 7.5.** For  $n_1, \dots, n_k \geq 0$ ,

$$(241) \quad \lambda^2 \frac{\partial}{\partial t_{n_1-1}} \dots \lambda^2 \frac{\partial}{\partial t_{n_k-1}} |0\rangle = \binom{n_1 + \dots + n_k}{n_1, \dots, n_k} \lambda^2 \frac{\partial}{\partial t_{n_1 + \dots + n_k - 1}} |0\rangle.$$

*Proof.*

$$\begin{aligned} & \lambda^2 \frac{\partial}{\partial t_{n_1-1}} \dots \lambda^2 \frac{\partial}{\partial t_{n_k-1}} |0\rangle = \frac{x^{n_1}}{n_1!} \dots \frac{x^{n_k}}{n_k!} |0\rangle \\ &= \binom{n_1 + \dots + n_k}{n_1, \dots, n_k} \frac{x^{n_1 + \dots + n_k}}{(n_1 + \dots + n_k)!} |0\rangle \\ &= \binom{n_1 + \dots + n_k}{n_1, \dots, n_k} \lambda^2 \frac{\partial}{\partial t_{n_1 + \dots + n_k - 1}} |0\rangle. \end{aligned}$$

$\square$

Next we have

**Lemma 7.6.**

$$(242) \quad \partial_x |0\rangle = \frac{1}{\lambda^2} \sum_{n=0}^{\infty} (t_n - \delta_{n,1}) \frac{\partial}{\partial t_{n-1}} |0\rangle,$$

$$(243) \quad \partial_x \frac{x^m}{m!} |0\rangle = \lambda^2 \frac{\partial}{\partial t_{m-2}} |0\rangle$$

$$(244) \quad \left\langle \frac{x^n}{n!} \right\rangle = \lambda^2 \frac{\partial}{\partial t_{n-1}} Z$$

**Theorem 7.7.** *The partition function of the topological 1D gravity satisfies the following equation:*

$$(245) \quad \lambda^2 \frac{\partial}{\partial t_{n_1-1}} \cdots \lambda^2 \frac{\partial}{\partial t_{n_k-1}} Z = \binom{n_1 + \cdots + n_k}{n_1, \dots, n_k} \lambda^2 \frac{\partial}{\partial t_{n_1 + \cdots + n_k - 1}} Z.$$

*In particular, in genus zero,*

$$(246) \quad \frac{\partial F_0}{\partial t_{n_1-1}} \cdots \frac{\partial F_0}{\partial t_{n_k-1}} = \binom{n_1 + \cdots + n_k}{n_1, \dots, n_k} \frac{\partial F_0}{\partial t_{n_1 + \cdots + n_k - 1}}.$$

*Proof.*

$$\begin{aligned} & \lambda^2 \frac{\partial}{\partial t_{n_1-1}} \cdots \lambda^2 \frac{\partial}{\partial t_{n_k-1}} Z = \left\langle \frac{x^{n_1}}{n_1!} \cdots \frac{x^{n_k}}{n_k!} \right\rangle \\ &= \binom{n_1 + \cdots + n_k}{n_1, \dots, n_k} \left\langle \frac{x^{n_1 + \cdots + n_k}}{(n_1 + \cdots + n_k)!} \right\rangle \\ &= \binom{n_1 + \cdots + n_k}{n_1, \dots, n_k} \lambda^2 \frac{\partial}{\partial t_{n_1 + \cdots + n_k - 1}} Z. \end{aligned}$$

□

**7.8. Another version of Virasoro constraints.** Recall we have derived that  $Z$  satisfies the Virasoro constraints with the following Virasoro operators:

$$(247) \quad L_{-1} = \frac{t_0}{\lambda^2} + \sum_{n \geq 1} (t_n - \delta_{n,1}) \frac{\partial}{\partial t_{n-1}},$$

$$(248) \quad L_0 = 1 + \sum_{m \geq 0} (t_m - \delta_{m,1}) (m+1) \frac{\partial}{\partial t_m},$$

$$(249) \quad L_m = \lambda^2 (m+1)! \frac{\partial}{\partial t_{m-1}} + \sum_{n \geq 0} (t_n - \delta_{n,1}) \frac{(m+n+1)!}{n!} \frac{\partial}{\partial t_{m+n}},$$

for  $m \geq 1$ . Now applying (245), we get:

**Theorem 7.8.** *The partition function  $Z$  of topological 1D gravity satisfies the following equations for  $m \geq -1$ :*

$$(250) \quad \tilde{L}_m Z = 0,$$

where

$$(251) \quad \tilde{L}_{-1} = \frac{t_0}{\lambda^2} + \sum_{m \geq 1} (t_m - \delta_{m,1}) \frac{\partial}{\partial t_{m-1}},$$

$$(252) \quad \tilde{L}_0 = 1 + \sum_{m \geq 0} (t_m - \delta_{m,1})(m+1) \frac{\partial}{\partial t_m},$$

$$(253) \quad \tilde{L}_1 = 2\lambda^2 \frac{\partial}{\partial t_0} + \sum_{n \geq 0} (t_n - \delta_{n,1}) \frac{(n+2)!}{n!} \frac{\partial}{\partial t_{n+1}},$$

$$(254) \quad \begin{aligned} \tilde{L}_m = & 2\lambda^2 m! \frac{\partial}{\partial t_{m-1}} + \lambda^4 \sum_{\substack{m_1+m_2=m \\ m_1, m_2 \geq 1}} m_1! \frac{\partial}{\partial t_{m_1-1}} \cdot m_2! \frac{\partial}{\partial t_{m_2-1}} \\ & + \sum_{n \geq 0} (t_n - \delta_{n,1}) \frac{(m+n+1)!}{n!} \frac{\partial}{\partial t_{m+n}}, \end{aligned}$$

for  $m \geq 2$ . Furthermore,  $\{\tilde{L}_m\}_{m \geq 1}$  satisfies the following commutation relations:

$$(255) \quad [\tilde{L}_m, \tilde{L}_n] = 0,$$

for  $m, n \geq -1$ .

*Proof.* When  $m \geq 2$ , for  $m_1 = 1, \dots, m-1$ ,  $m_2 = m - m_1$ ,

$$m! \lambda^2 \frac{\partial}{\partial t_{m-1}} Z = \lambda^2 m_1! \frac{\partial}{\partial t_{m_1-1}} \cdot \frac{\partial}{\partial t_{m_2-1}} Z,$$

it follows that

$$\begin{aligned} (m+1)! \lambda^2 \frac{\partial}{\partial t_{m-1}} Z &= 2 \cdot m! \lambda^2 \frac{\partial}{\partial t_{m-1}} Z \\ &+ \sum_{\substack{m_1+m_2=m \\ m_1, m_2 \geq 1}} \lambda^2 m_1! \frac{\partial}{\partial t_{m_1-1}} \cdot \frac{\partial}{\partial t_{m_2-1}} Z. \end{aligned}$$

Therefore one can derive (250) from (181). The commutation relations can be checked by a standard calculation.  $\square$



**Corollary 7.9.** *The genus zero free energy  $F_0$  of the topological 1D gravity satisfies the following equations:*

$$(256) \quad \frac{\partial F_0}{\partial t_0} = t_0 + \sum_{m \geq 1} t_m \frac{\partial F_0}{\partial t_{m-1}},$$

$$(257) \quad (m+2)! \frac{\partial F_0}{\partial t_{m+1}} = \sum_{n \geq 0} \frac{(m+n+1)!}{n!} t_n \frac{\partial F_0}{\partial t_{m+n}},$$

where  $m \geq 0$ .

## 8. N-POINT FUNCTIONS IN TOPOLOGICAL 1D GRAVITY

We will compute  $n$ -point functions in this Section. We also present recursion relations satisfied by them. Our technical tools are the loop operators.

**8.1. Two kinds of  $n$ -point functions.** In this subsection, we define two kinds of  $n$ -point functions of the topological 1D gravity:

$$\begin{aligned} \hat{W}_{g,n}(w_1, \dots, w_n; \mathbf{t}) &= \delta_{g,0} \delta_{n,1} + \sum_{m_1, \dots, m_n \geq 0} \frac{\partial^n F_g}{\partial t_{m_1} \cdots \partial t_{m_n}} w_1^{m_1} \cdots w_n^{m_n}, \\ W_{g,n}(z_1, \dots, z_n; \mathbf{t}) &= \frac{\delta_{g,0} \delta_{n,1}}{z_1} + \sum_{m_1, \dots, m_n \geq 1} \frac{\partial^n F_g}{\partial t_{m_1-1} \cdots \partial t_{m_n-1}} \prod_{j=1}^n m_j! z_j^{-m_j-1}. \end{aligned}$$

They are related to each other by Laplace transform:

$$(258) \quad W_{g,n}(z_1, \dots, z_n; \mathbf{t}) = \int_{\mathbb{R}_+^n} e^{-\sum_{j=1}^n w_j z_j} \hat{W}_{g,n}(w_1, \dots, w_n; \mathbf{t}) dw_1 \cdots dw_n,$$

where  $\mathbb{R}_+ = [0, +\infty)$ .

Define

$$\begin{aligned} \hat{W}_n(w_1, \dots, w_n; \mathbf{t}) &= \sum_{g \geq 0} \lambda^{2g} \hat{W}_{g,n}(w_1, \dots, w_n; \mathbf{t}), \\ W_n(z_1, \dots, z_n; \mathbf{t}) &= \sum_{g \geq 0} \lambda^{2g} W_{g,n}(z_1, \dots, z_n; \mathbf{t}). \end{aligned}$$

It is clear that

$$\begin{aligned} \hat{W}_n(w_1, \dots, w_n; \mathbf{t}) &= \delta_{g,0} \delta_{n,1} + \lambda^2 \sum_{m_1, \dots, m_n \geq 0} \frac{\partial^n F}{\partial t_{m_1} \cdots \partial t_{m_n}} w_1^{m_1} \cdots w_n^{m_n}, \\ W_n(z_1, \dots, z_n; \mathbf{t}) &= \frac{\delta_{g,0} \delta_{n,1}}{z_1} + \lambda^2 \sum_{m_1, \dots, m_n \geq 1} \frac{\partial^n F}{\partial t_{m_1-1} \cdots \partial t_{m_n-1}} \prod_{j=1}^n m_j! z_j^{-m_j-1}. \end{aligned}$$

**8.2. Loop operators.** Similarly, define two kinds of loop operators:

$$\begin{aligned}\hat{B}(w) &= \sum_{n \geq 0} w^n \frac{\partial}{\partial t_n}, \\ B(z) &= \sum_{m \geq 1} \frac{m!}{z^{m+1}} \frac{\partial}{\partial t_{m-1}}.\end{aligned}$$

They are related by a Laplace transform, and it is clear that for  $n \geq 1$ ,

$$(259) \quad \hat{W}_n(w_1, \dots, w_n; \mathbf{t}) = \delta_{n,1} + \hat{B}(w) \hat{W}_{n-1}(w_2, \dots, w_n; \mathbf{t}),$$

$$(260) \quad W_n(z_1, \dots, z_n; \mathbf{t}) = \frac{\delta_{n,1}}{z_1} + B(z_1) W_{n-1}(z_2, \dots, z_n; \mathbf{t}).$$

Here

$$(261) \quad \hat{W}_0(\mathbf{t}) = W_0(\mathbf{t}) = \lambda^2 F.$$

**8.3. Genus zero one-point functions.** By (154), in genus zero we have:

$$(262) \quad \frac{\partial F_0}{\partial t_m} = \frac{1}{(m+1)!} \left( \frac{\partial F_0}{\partial t_0} \right)^{m+1} = \frac{1}{(m+1)!} I_0^{m+1}.$$

It follows that

$$(263) \quad \hat{W}_{0,1}(w_1; \mathbf{t}) = e^{w_1 \frac{\partial F_0}{\partial t_0}} = e^{w_1 I_0},$$

and

$$(264) \quad W_{0,1}(z_1; \mathbf{t}) = \frac{1}{z_1 - \frac{\partial F_0}{\partial t_0}} = \frac{1}{z_1 - I_0}.$$

In particular,

$$(265) \quad \frac{\partial F_0}{\partial t_n}(t_0) = \frac{t_0^{n+1}}{(n+1)!},$$

and so we have

$$(266) \quad \hat{W}_{0,1}(w_1; t_0) = e^{t_0 w_1}$$

and

$$(267) \quad W_{0,1}(z_1; t_0) = \frac{1}{z_1 - t_0}.$$

8.4. **One-point function in arbitrary genera.** By (154),

$$(268) \quad \frac{\partial F}{\partial t_n} = \frac{\lambda^{2n}}{(n+1)!} \left( \frac{\partial}{\partial t_0} + \frac{\partial F}{\partial t_0} \right)^{n+1} 1.$$

one gets:

$$(269) \quad \hat{W}_1(w; \mathbf{t}) = e^{\lambda^2 w (\frac{\partial}{\partial t_0} + \frac{\partial F}{\partial t_0})} 1$$

and

$$(270) \quad W_1(z; \mathbf{t}) = \frac{1}{z - \lambda^2 (\frac{\partial}{\partial t_0} + \frac{\partial F}{\partial t_0})} 1.$$

When restricted to the  $t_0$ -line, i.e., take  $t_n = 0$  for  $n \geq 1$ ,

$$\begin{aligned} \frac{\partial F}{\partial t_0}(t_0) &= \frac{t_0}{\lambda^2}, \\ \frac{\partial F}{\partial t_1}(t_0) &= \frac{1}{2!} \left( \frac{t_0^2}{\lambda^2} + 1 \right), \\ \frac{\partial F}{\partial t_2}(t_0) &= \frac{1}{3!} \left( \frac{t_0^3}{\lambda^2} + 3t_0 \right), \\ \frac{\partial F}{\partial t_3}(t_0) &= \frac{1}{4!} \left( \frac{1}{\lambda^2} t_0^4 + 6t_0^2 + 3\lambda^2 \right), \\ \frac{\partial F}{\partial t_4}(t_0) &= \frac{1}{5!} \left( \frac{1}{\lambda^2} t_0^5 + 10t_0^3 + 15\lambda^2 t_0 \right). \end{aligned}$$

These are essentially Hermite polynomials!

The coefficients turns out to give the triangle of Bessel numbers

$$(271) \quad T(n, k) = \frac{n!}{(n-2k)!k!2^k},$$

they have the following exponential generating series:

$$(272) \quad \sum_{n,k} \frac{1}{n!} T_{n,k} t^n z^k = \exp(z + tz^2/2).$$

In fact, we have

**Theorem 8.1.** *The following formula holds:*

$$(273) \quad \frac{\partial F}{\partial t_n}(t_0) = \frac{1}{(n+1)!} \frac{1}{\lambda^2} \sum_{k=0}^{[n/2]} T(n+1, k) t_0^{n+1-2k} \lambda^{2k}.$$

*Proof.* By (269) and  $\frac{\partial F}{\partial t_0}(t_0) = \frac{t_0}{\lambda^2}$ ,

$$\begin{aligned} 1 + \lambda^2 \sum_{n \geq 1} w^n \frac{\partial F}{\partial t_{n-1}}(t_0) &= e^{\lambda^2 w (\frac{\partial}{\partial t_0} + \frac{t_0}{\lambda^2})} 1 = e^{\frac{1}{2} w^2 \lambda^2} e^{w t_0} e^{w \lambda^2 \frac{\partial}{\partial t_0}} 1 \\ &= e^{\frac{1}{2} w^2 \lambda^2 + w t_0}. \end{aligned}$$

In the above we have used the Campbell-Baker-Hausdorff formula:

$$e^X e^Y = \exp(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \dots)$$

for  $X = \frac{t_0}{\lambda^2}$ ,  $Y = w \frac{\partial}{\partial t_0}$ . The proof is completed by (272).  $\square$

**Corollary 8.2.** *When restricted to the  $t_0$ -line, the one-point function of topological 1D gravity is given by:*

$$(274) \quad W_1(z; t_0) = \sum_{g=0}^{\infty} \frac{(2g-1)!!}{(z-t_0)^{2g+1}} \lambda^{2g}.$$

Since the generating series of the double factorials can be written as a continued fraction:

$$(275) \quad \sum_{n \geq 0} (2n-1)!! x^n = 1 / (1 - x / (1 - 2x / (1 - 3x / (1 - 4x / (1 - \dots),$$

one can express the genus zero one-point function restricted to the  $t_0$ -line by continued fraction:

$$(276) \quad \begin{aligned} &W_1(z; t_0) \\ &= \frac{1}{z-t_0} \cdot 1 / (1 - \frac{\lambda^2}{(z-t_0)^2} / (1 - 2 \frac{\lambda^2}{(z-t_0)^2} / (1 - 3 \frac{\lambda^2}{(z-t_0)^2} / (1 - \dots \end{aligned}$$

As a strange coincidence, the Bessel numbers also appeared in the author's study of topological 2D gravity.

**Lemma 8.3.** *The following identity holds:*

$$(277) \quad \exp(t \frac{\partial}{\partial x}) \exp(f(x)) = \exp \sum_{n \geq 0} \frac{t^n}{n!} \frac{\partial^n f(x)}{\partial x^n}.$$

*Proof.* We first check that both sides of the identity satisfies the same differential equation:

$$(278) \quad \frac{\partial}{\partial t} G(t, x) = \frac{\partial}{\partial x} G(t, x),$$

and they have the same initial value  $G(0, x) = e^{f(x)}$ . Write

$$G(t, x) = \sum_{n \geq 0} G_n(x) \frac{t^n}{n!},$$

then above differential equation is equivalent to the following recursion relations:

$$(279) \quad G_n(x) = \frac{\partial}{\partial x} G_{n-1}(x).$$

This completes the proof.  $\square$

**Theorem 8.4.** *For topological 1D gravity, we have*

$$(280) \quad \hat{W}_1(w; \mathbf{t}) = \exp \sum_{n=1}^{\infty} \frac{w^n \lambda^{2n}}{n!} \frac{\partial^n F}{\partial t_0^n}$$

and

$$(281) \quad W_1(z; \mathbf{t}) = \frac{1}{z} \sum_{m_1, \dots, m_n \geq 0} \frac{(\sum_{j=1}^n j m_j)!}{\prod_{j=1}^n m_j!} \prod_{j=1}^n \left( \frac{\lambda^{2j}}{z^j j!} \frac{\partial^j F}{\partial t_0^j} \right)^{m_j}.$$

*Proof.* Note

$$\frac{\partial F}{\partial t_{n-1}} = \frac{1}{Z} \frac{\partial Z}{\partial t_{n-1}} = \frac{1}{Z} \cdot \frac{\lambda^{2n-2}}{n!} \frac{\partial^n Z}{\partial t_0^n},$$

therefore,

$$\begin{aligned} \hat{W}_1(w) &= 1 + \sum_{n \geq 1} w^n \lambda^2 \frac{\partial F}{\partial t_{n-1}} = 1 + \frac{1}{Z} \sum_{n \geq 1} \frac{w^n \lambda^{2n}}{n!} \frac{\partial^n Z}{\partial t_0^n} \\ &= \exp(-F) \cdot \exp(w \lambda^2 \frac{\partial}{\partial t_0}) \exp(F) \\ &= \exp \sum_{n=1}^{\infty} \frac{w^n \lambda^{2n}}{n!} \frac{\partial^n F}{\partial t_0^n}. \end{aligned}$$

This proves (280). By expanding the right-hand side of (280) as a power series in  $w$  and take the Laplace transform, one gets (281).  $\square$

*Remark 8.5.* Note (280) and (281) can be rewritten in terms of loop operators as follows:

$$(282) \quad 1 + \hat{B}(w)F = \exp \sum_{n=1}^{\infty} \frac{w^n \lambda^{2n}}{n!} \frac{\partial^n F}{\partial t_0^n}$$

$$(283) \quad \frac{1}{z} + B(z)F = \frac{1}{z} \sum_{m_1, \dots, m_n \geq 0} \frac{(\sum_{j=1}^n j m_j)!}{\prod_{j=1}^n m_j!} \prod_{j=1}^n \left( \frac{\lambda^{2j}}{z^j j!} \frac{\partial^j F}{\partial t_0^j} \right)^{m_j}.$$

8.5. **Genus zero two-point functions.** When  $n_1, n_2 \geq 0$ ,

$$\begin{aligned}
\frac{\partial^2 F_0}{\partial t_{n_1} \partial t_{n_2}} &= \frac{\partial}{\partial t_{n_1}} \frac{1}{(n_2 + 1)!} \left( \frac{\partial F_0}{\partial t_0} \right)^{n_1+1} \\
&= \frac{\partial}{\partial t_{n_1}} \frac{1}{n_2!} \left( \frac{\partial F_0}{\partial t_0} \right)^{n_1} \cdot \frac{\partial}{\partial t_0} \frac{\partial F_0}{\partial t_{n_1}} \\
&= \frac{1}{n_1! n_2!} \left( \frac{\partial F_0}{\partial t_0} \right)^{n_1+n_2} \cdot \frac{\partial^2 F_0}{\partial t_0^2} \\
&= \frac{1}{n_1! n_2! (n_1 + n_2 + 1)} \frac{\partial}{\partial t_0} \left( \frac{\partial F_0}{\partial t_0} \right)^{n_1+n_2+1}.
\end{aligned}$$

It follows that

$$(284) \quad \sum_{n_1, n_2 \geq 0} z_1^{n_1} z_2^{n_2} \frac{\partial^2 F_0}{\partial t_{n_1} \partial t_{n_2}} = \frac{\partial^2 F_0}{\partial t_0^2} \cdot e^{(z_1+z_2) \frac{\partial F_0}{\partial t_0}} = \frac{\partial}{\partial t_0} \frac{e^{(z_1+z_2) \frac{\partial F_0}{\partial t_0}}}{z_1 + z_2},$$

and

$$\begin{aligned}
W_{0,2}(z_1, z_2) &= \sum_{n_1, n_2 \geq 0} \frac{n_1 + 1}{z_1^{n_1+2}} \frac{n_2 + 1}{z_2^{n_2+2}} \left( \frac{\partial F_0}{\partial t_0} \right)^{n_1+n_2} \cdot \frac{\partial^2 F_0}{\partial t_0^2} \\
&= \frac{1}{(z_1 - \frac{\partial F_0}{\partial t_0})^2} \frac{1}{(z_2 - \frac{\partial F_0}{\partial t_0})^2} \frac{\partial^2 F_0}{\partial t_0^2} \\
&= \partial_{t_0} W_{0,1}(z_1) \cdot \partial_{t_0} W_{0,1}(z_2) \cdot \frac{1}{\frac{\partial^2 F_0}{\partial t_0^2}}.
\end{aligned}$$

In particular, when restricted to the  $t_0$ -line,

$$(285) \quad \frac{\partial^2 F_0}{\partial t_{n_1} \partial t_{n_2}}(t_0) = \frac{t_0^{n_1+n_2}}{n_1! n_2!},$$

and so we have

$$(286) \quad \hat{W}_{0,2}(w_1, w_2; t_0) = e^{t_0(z_1+z_2)},$$

and

$$(287) \quad W_{0,2}(z_1, z_2; t_0) = \frac{1}{(z_1 - t_0)^2 (z_2 - t_0)^2}.$$

We also have

$$(288) \quad \sum_{n_1, n_2 \geq 0} \frac{n_1!}{z_1^{n_1+1}} \frac{n_2!}{z_2^{n_2+1}} \frac{\partial^2 F_0}{\partial t_{n_1} \partial t_{n_2}}(t_0) = \frac{1}{(z_1 - t_0)(z_2 - t_0)}.$$

### 8.6. Two-point function in arbitrary genera.

**Theorem 8.6.** *The two-point function  $\hat{W}_2(w_1, w_2; \mathbf{t})$  of the topological 1D gravity is given by the following formula:*

$$(289) \quad \begin{aligned} \lambda^2 \hat{W}_{0,2}(w_1, w_2; \mathbf{t}) = & \exp\left(\sum_{n=1}^{\infty} \frac{(w_1 + w_2)^n \lambda^{2n}}{n!} \frac{\partial^n F}{\partial t_0^n}\right) \\ & - \exp\left(\sum_{n=1}^{\infty} \frac{w_2^n \lambda^{2n}}{n!} \frac{\partial^n F}{\partial t_0^n}\right) \cdot \exp\left(\sum_{n=1}^{\infty} \frac{w_1^n \lambda^{2n}}{n!} \frac{\partial^n F}{\partial t_0^n}\right). \end{aligned}$$

*Proof.* This can be proved by a direct calculation as follows:

$$\begin{aligned} & \lambda^2 \hat{W}(w_1, w_2; \mathbf{t}) \\ = & \lambda^4 \sum_{n_1, n_2 \geq 1} w_1^{n_1} w_2^{n_2} \frac{\partial^2 F}{\partial t_{n_1-1} \partial t_{n_2-1}} \\ = & \lambda^2 \sum_{n_1 \geq 1} w_1^{n_1} \frac{\partial}{\partial t_{n_1-1}} \exp \sum_{n=1}^{\infty} \frac{w_2^n \lambda^{2n}}{n!} \frac{\partial^n F}{\partial t_0^n} \\ = & \exp\left(\sum_{n=1}^{\infty} \frac{w_2^n \lambda^{2n}}{n!} \frac{\partial^n F}{\partial t_0^n}\right) \cdot \lambda^2 \sum_{n_1 \geq 1} w_1^{n_1} \frac{\partial}{\partial t_{n_1-1}} \sum_{n=1}^{\infty} \frac{w_2^n \lambda^{2n}}{n!} \frac{\partial^n F}{\partial t_0^n} \\ = & \exp\left(\sum_{n=1}^{\infty} \frac{w_2^n \lambda^{2n}}{n!} \frac{\partial^n F}{\partial t_0^n}\right) \cdot \lambda^2 \sum_{n_1 \geq 1} w_1^{n_1} \frac{\partial}{\partial t_{n_1-1}} \sum_{n=1}^{\infty} \frac{w_2^n \lambda^{2n}}{n!} \frac{\partial^n F}{\partial t_0^n} \\ = & \exp\left(\sum_{n=1}^{\infty} \frac{w_2^n \lambda^{2n}}{n!} \frac{\partial^n F}{\partial t_0^n}\right) \cdot \sum_{n \geq 1} \frac{w_2^n \lambda^{2n}}{n!} \frac{\partial^n}{\partial t_0^n} \left( \sum_{n_1 \geq 1} \frac{\lambda^2 w_1^{n_1}}{n_1!} \frac{\partial F}{\partial t_{n_1-1}} \right) \\ = & \exp\left(\sum_{n=1}^{\infty} \frac{w_2^n \lambda^{2n}}{n!} \frac{\partial^n F}{\partial t_0^n}\right) \cdot \sum_{n \geq 1} \frac{w_2^n \lambda^{2n}}{n!} \frac{\partial^n}{\partial t_0^n} \exp\left(\sum_{n=1}^{\infty} \frac{w_1^n \lambda^{2n}}{n!} \frac{\partial^n F}{\partial t_0^n}\right) \\ = & \exp\left(\sum_{n=1}^{\infty} \frac{w_2^n \lambda^{2n}}{n!} \frac{\partial^n F}{\partial t_0^n}\right) \cdot \exp\left(\sum_{n=1}^{\infty} \frac{w_1^n \lambda^{2n}}{n!} \frac{\partial^n F}{\partial t_0^n}\right) \\ & \cdot \left( \exp\left(\sum_{m,n=1}^{\infty} \frac{w_1^n \lambda^{2n}}{n!} \frac{w_2^m \lambda^{2m}}{m!} \frac{\partial^{m+n} F}{\partial t_0^{m+n}}\right) - 1 \right) \\ = & \exp\left(\sum_{n=1}^{\infty} \frac{(w_1 + w_2)^n \lambda^{2n}}{n!} \frac{\partial^n F}{\partial t_0^n}\right) \\ & - \exp\left(\sum_{n=1}^{\infty} \frac{w_2^n \lambda^{2n}}{n!} \frac{\partial^n F}{\partial t_0^n}\right) \cdot \exp\left(\sum_{n=1}^{\infty} \frac{w_1^n \lambda^{2n}}{n!} \frac{\partial^n F}{\partial t_0^n}\right). \end{aligned}$$

□

*Remark 8.7.* A more conceptual way to rewrite (289) is as follows:

$$(290) \quad \lambda^2 \hat{W}_{0,2}(w_1, w_2; \mathbf{t}) = \hat{W}_{0,1}(w_1 + w_2; \mathbf{t}) - \hat{W}_{0,1}(w_1; \mathbf{t}) \cdot \hat{W}_{0,1}(w_2; \mathbf{t}).$$

It will be useful to note this is equivalent to

$$(291) \quad \lambda^2 \hat{B}(w_1) \hat{W}_{0,1}(z_2; \mathbf{t}) = \hat{W}_{0,1}(w_1 + w_2; \mathbf{t}) - \hat{W}_{0,1}(w_1; \mathbf{t}) \cdot \hat{W}_{0,1}(w_2; \mathbf{t}).$$

**Corollary 8.8.** *When restricted to the  $t_0$ -line, the two-point function  $\hat{W}_2(w_1, w_2, \mathbf{t})$  is given by the following formula:*

$$(292) \quad \begin{aligned} \lambda^2 \hat{W}(w_1, w_2; t_0) = & \exp\left((w_1 + w_2)t_0 + \frac{(w_1 + w_2)^2 \lambda^2}{2!}\right) \\ & - \exp\left(w_1 t_0 + \frac{w_1^2 \lambda^2}{2!}\right) \cdot \exp\left(w_2 t_0 + \frac{w_2^2 \lambda^2}{2!}\right). \end{aligned}$$

One can compute  $W_2(z_1, z_2; \mathbf{t})$  by Laplace transform the formula for  $\hat{W}_2(w_1, w_2; \mathbf{t})$ . Here we will use a more conceptual method by applying the loop operator  $B(z_1)$  on  $W_1(z_2; \mathbf{t})$ .

**Theorem 8.9.** *The two-point function  $W_2(z_1, z_2; \mathbf{t})$  of the topological 1D gravity is given by the following formula:*

$$(293) \quad \lambda^2 W_2(z_1, z_2) = \sum_{k \geq 1} \frac{(-1)^k \lambda^{2k}}{k!} \frac{\partial^k}{\partial z^k} W_1(z_2; \mathbf{t}) \cdot \frac{\partial^k}{\partial t_0^k} W_1(z_1; \mathbf{t}).$$

*Proof.* Note  $B(z_1)$  is a derivation, therefore, by (281),

$$\begin{aligned} W_2(z_1, z_2) &= B(z_1) W_1(z_2; \mathbf{t}) \\ &= B(z_1) \frac{1}{z_2} \sum_{m_1, \dots, m_n \geq 0} \frac{(\sum_{j=1}^n j m_j)!}{\prod_{j=1}^n m_j!} \prod_{j=1}^n \left( \frac{\lambda^{2j}}{z_2^j j!} \frac{\partial^j F}{\partial t_0^j} \right)^{m_j} \\ &= \frac{1}{z_2} \sum_{m_1, \dots, m_n \geq 0} \frac{(\sum_{j=1}^n j m_j)!}{\prod_{j=1}^n m_j!} \sum_{k=1}^n \prod_{j=1}^n \left( \frac{\lambda^{2j}}{z_2^j j!} \frac{\partial^j F}{\partial t_0^j} \right)^{m_j - \delta_{j,k}} \\ &\quad \cdot m_k \cdot \frac{\partial^k}{\partial t_0^k} B(z_1) F \\ &= \sum_{k \geq 1} \frac{(-1)^k \lambda^{2k}}{k!} \frac{\partial^k}{\partial z^k} \left( \frac{1}{z_2} \sum_{m_1, \dots, m_n \geq 0} \frac{(\sum_{j=1}^n j m_j)!}{\prod_{j=1}^n m_j!} \prod_{j=1}^n \left( \frac{\lambda^{2j}}{z_2^j j!} \frac{\partial^j F}{\partial t_0^j} \right)^{m_j} \right) \\ &\quad \cdot \frac{\partial^k}{\partial t_0^k} B(z_1) F \\ &= \lambda^{-2} \sum_{k \geq 1} \frac{(-1)^k \lambda^{2k}}{k!} \frac{\partial^k}{\partial z^k} W_1(z_2; \mathbf{t}) \cdot \frac{\partial^k}{\partial t_0^k} W_1(z_1; \mathbf{t}). \end{aligned}$$

□



*Remark 8.10.* Note (293) can be rewritten as follows:

$$(294) \quad B(z_1)W_1(z_2; \mathbf{t}) = \sum_{k \geq 1} \frac{(-1)^k \lambda^{2k}}{k!} \frac{\partial^k}{\partial z^k} W_1(z_2; \mathbf{t}) \cdot \frac{\partial^k}{\partial t_0^k} W_1(z_1; \mathbf{t}).$$

One can also symmetrize this:

$$(295) \quad B(z_1)W_1(z_2; \mathbf{t}) = \frac{1}{2} \sum_{k \geq 1} \frac{(-1)^k \lambda^{2k}}{k!} \left( \frac{\partial^k}{\partial z^k} W_1(z_1; \mathbf{t}) \cdot \frac{\partial^k}{\partial t_0^k} W_1(z_2; \mathbf{t}) + \frac{\partial^k}{\partial z^k} W_1(z_2; \mathbf{t}) \cdot \frac{\partial^k}{\partial t_0^k} W_1(z_1; \mathbf{t}) \right).$$

**Corollary 8.11.** *When restricted to the  $t_0$ -line, the two-point function  $W_2(z_1, z_2, \mathbf{t})$  is given by the following formula:*

$$(296) \quad W(z_1, z_2; t_0) = \sum_{k \geq 1} \frac{\lambda^{2k}}{k!} \sum_{g=0}^{\infty} \frac{(2g-1)!! \prod_{j=0}^{k-1} (2g+1+j)}{(z_1 - t_0)^{2g+1+k}} \lambda^{2g} \cdot \sum_{g=0}^{\infty} \frac{(2g-1)!! \prod_{j=0}^{k-1} (2g+1+j)}{(z_2 - t_0)^{2g+1+k}} \lambda^{2g}.$$

**8.7. Genus zero  $l$ -point functions.** Inductively we find for  $l \geq 3$ :

$$\begin{aligned} \frac{\partial^l F_0}{\partial t_{n_1} \cdots \partial t_{n_l}} &= \frac{1}{n_1! \cdots n_l! (n_1 + \cdots + n_l + 1)} \frac{\partial^{l-1}}{\partial t_0^{l-1}} \left( \frac{\partial F_0}{\partial t_0} \right)^{n_1 + \cdots + n_l + 1} \\ &= \frac{1}{n_1! \cdots n_l! \partial t_0^{l-2}} \left[ \left( \frac{\partial F_0}{\partial t_0} \right)^{n_1 + \cdots + n_l} \cdot \frac{\partial^2 F_0}{\partial t_0^2} \right]. \end{aligned}$$

It follows that

$$(297) \quad \hat{W}_{0,l}(w_1, \dots, w_l; \mathbf{t}) = \frac{\partial^{l-1}}{\partial t_0^{l-1}} \frac{e^{\frac{\partial F_0}{\partial t_0} \sum_{j=1}^l w_j}}{\sum_{j=1}^l w_j},$$

and

$$(298) \quad W_{0,l}(z_1, \dots, z_l; \mathbf{t}) = \frac{\partial^{l-2}}{\partial t_0^{l-2}} \left( \frac{1}{(z_1 - \frac{\partial F_0}{\partial t_0})^2} \cdots \frac{1}{(z_l - \frac{\partial F_0}{\partial t_0})^2} \frac{\partial^2 F_0}{\partial t_0^2} \right).$$

In particular, when restricted to the  $t_0$ -line,

$$(299) \quad \frac{\partial^l F_0}{\partial t_{n_1} \cdots \partial t_{n_l}}(t_0) = \frac{\partial^{l-2}}{\partial t_0^{l-2}} \frac{t_0^{n_1 + \cdots + n_l}}{n_1! \cdots n_l!},$$

and so

$$(300) \quad \hat{W}_{0,l}(w_1, \dots, w_l; t_0) = e^{t_0 \sum_{j=1}^l w_j} \left( \sum_{j=1}^l w_j \right)^{l-2}.$$

and

$$(301) \quad W_{0,l}(z_1, \dots, z_l; t_0) = \frac{\partial^{l-2}}{\partial t_0^{l-2}} \frac{1}{\prod_{j=1}^l (z_j - t_0)^2}.$$

We also have

$$(302) \quad \sum_{n_1, \dots, n_l \geq 0} \prod_{j=1}^l \frac{n_j!}{z_j^{n_j+1}} \cdot \frac{\partial^l F_0}{\partial t_{n_1} \dots \partial t_{n_l}}(t_0) = \frac{\partial^{l-2}}{\partial t_0^{l-2}} \frac{1}{\prod_{j=1}^l (z_j - t_0)^2}.$$

By (299), one can derive the following formula:

$$(303) \quad \sum_{n_1, \dots, n_l \geq 0} \langle \tau_{n_1} \dots \tau_{n_l} \rangle_0 \cdot \prod_{j=1}^l x_j^{n_j} = (x_1 + \dots + x_l)^{l-2}.$$

**8.8. General  $l$ -point functions in arbitrary genera.** We now apply (259) and (291) repeatedly to compute  $\hat{W}_l$ . Let us fix some notations. By  $[l]$  we mean the set of indices  $\{1, \dots, l\}$ . By  $I_1 \coprod \dots \coprod I_k = [l]$  we mean a partition of  $[l]$  into disjoint union of  $k$  nonempty subsets  $I_1, \dots, I_k$ . Given such a partition, for  $j = 1, \dots, k$ ,  $|I_j|$  denotes the number of elements in  $I_j$ , and we define

$$w_{I_j} = \sum_{i \in I_j} w_i.$$

**Theorem 8.12.** *The  $l$ -point function  $\hat{W}_l(w_1, \dots, w_l; \mathbf{t})$  of the topological 1D gravity is given by the following formula:*

$$(304) \quad \lambda^{2l-2} \hat{W}(w_1, \dots, w_l; \mathbf{t}) = \sum_{I_1 \coprod \dots \coprod I_k = [l]} (-1)^{k-1} k! \prod_{j=1}^k W_{|I_j|}(w_{I_j}; \mathbf{t}).$$

*Proof.* This can be easily proved by induction on  $l$ . □

Similarly, one can apply (260) and (294) repeatedly to compute  $W_l$ . For example, from

$$\lambda^2 W_2(z_2, z_3; \mathbf{t}) = \sum_{k \geq 1} \frac{(-1)^k \lambda^{2k}}{k!} \frac{\partial^k}{\partial z^k} W_1(z_2; \mathbf{t}) \cdot \frac{\partial^k}{\partial t_0^k} W_1(z_3; \mathbf{t}),$$

one gets

$$\begin{aligned}
& \lambda^4 W_3(z_1, z_2, z_3; \mathbf{t}) \\
&= \lambda^2 B(z_1) \sum_{k \geq 1} \frac{(-1)^k \lambda^{2k}}{k!} \frac{\partial^k}{\partial z^k} W_1(z_2; \mathbf{t}) \cdot \frac{\partial^k}{\partial t_0^k} W_1(z_3; \mathbf{t}) \\
&= \sum_{k \geq 1} \frac{(-1)^k \lambda^{2k}}{k!} \frac{\partial^k}{\partial z^k} \lambda^2 B(z_1) W_1(z_2; \mathbf{t}) \cdot \frac{\partial^k}{\partial t_0^k} W_1(z_3; \mathbf{t}) \\
&+ \sum_{k \geq 1} \frac{(-1)^k \lambda^{2k}}{k!} \frac{\partial^k}{\partial z^k} W_1(z_2; \mathbf{t}) \cdot \frac{\partial^k}{\partial t_0^k} \lambda^2 B(z_1) W_1(z_3; \mathbf{t}) \\
&= \sum_{k \geq 1} \frac{(-1)^k \lambda^{2k}}{k!} \frac{\partial^k}{\partial z^k} \lambda^2 W_2(z_1, z_2; \mathbf{t}) \cdot \frac{\partial^k}{\partial t_0^k} W_1(z_3; \mathbf{t}) \\
&+ \sum_{k \geq 1} \frac{(-1)^k \lambda^{2k}}{k!} \frac{\partial^k}{\partial z^k} W_1(z_2; \mathbf{t}) \cdot \frac{\partial^k}{\partial t_0^k} \lambda^2 W_2(z_1, z_3; \mathbf{t}).
\end{aligned}$$

The result is not manifestly symmetric with respect to  $z_1, z_2, z_3$ , one can partially symmetrize it with respect to  $z_2, z_3$  to get:

$$\begin{aligned}
& \lambda^4 W_3(z_1, z_2, z_3; \mathbf{t}) \\
&= \frac{1}{2!} \sum_{I \sqcup J = [3]_1} \sum_{k \geq 1} \frac{(-1)^k \lambda^{2k}}{k!} \left( \frac{\partial^k}{\partial z^k} \lambda^2 W_2(z_1, z_I; \mathbf{t}) \cdot \frac{\partial^k}{\partial t_0^k} W_1(z_J; \mathbf{t}) \right. \\
&+ \left. \frac{\partial^k}{\partial z^k} W_1(z_I; \mathbf{t}) \cdot \frac{\partial^k}{\partial t_0^k} \lambda^2 W_2(z_1, z_J; \mathbf{t}) \right).
\end{aligned}$$

Here we use the following notation: For  $1 \leq i \leq n$ ,  $[n]_i = \{1, \dots, n\} - \{i\}$ . One can also symmetrize with respect to all three variables to get:

$$\begin{aligned}
& \lambda^4 W_3(z_1, z_2, z_3; \mathbf{t}) \\
&= \frac{1}{3!} \sum_{I \sqcup J = [3]} \sum_{k \geq 1} \frac{(-1)^k \lambda^{2k}}{k!} \frac{\partial^k}{\partial z^k} |I|! W_{|I|}(z_I; \mathbf{t}) \cdot \frac{\partial^k}{\partial t_0^k} |J|! W_{|J|}(z_J; \mathbf{t}).
\end{aligned}$$

By induction we get the following:

**Theorem 8.13.** *The  $l$ -point functions  $W_l$  of the topological 1D gravity satisfies the following recursion relations:*

$$\begin{aligned}
& \lambda^{2l-2} W_l(z_1, \dots, z_l; \mathbf{t}) \\
&= \frac{1}{l!} \sum_{I \amalg J = [l]} \sum_{k \geq 1} \frac{(-1)^k \lambda^{2k}}{k!} \binom{l-2}{|I|-1} \\
& \quad \cdot \frac{\partial^k}{\partial z^k} |I|! W_{|I|}(z_I; \mathbf{t}) \cdot \frac{\partial^k}{\partial t_0^k} |J|! W_{|J|}(z_J; \mathbf{t})
\end{aligned}$$

and

$$\begin{aligned}
& \lambda^{2l-2} W_l(z_1, \dots, z_l; \mathbf{t}) \\
&= \frac{1}{(l-1)!} \sum_{I \amalg J = [l-1]} \sum_{k \geq 1} \frac{(-1)^k \lambda^{2k}}{k!} \binom{l-3}{|I|-1} \\
& \quad \cdot \left( \frac{\partial^k}{\partial z^k} |I|! W_{|I|}(z_1, z_I; \mathbf{t}) \cdot \frac{\partial^k}{\partial t_0^k} |J|! W_{|J|}(z_J; \mathbf{t}) \right. \\
& \quad \left. + \frac{\partial^k}{\partial z^k} |I|! W_{|I|}(z_I; \mathbf{t}) \cdot \frac{\partial^k}{\partial t_0^k} |J|! W_{|J|}(z_1, z_J; \mathbf{t}) \right).
\end{aligned}$$

## 9. FEYNMAN RULES FOR $N$ -POINT FUNCTIONS

In this Section we discuss the Feynman rules for  $n$ -point functions in topological 1D gravity.

**9.1. Feynman rules for genus zero  $n$ -point functions.** We have already shown that

$$(305) \quad W_{0,n}(z_1, \dots, z_n; \mathbf{t}) = \frac{\partial^{n-2}}{\partial t_0^{n-2}} \left[ \frac{1}{(z_1 - \frac{\partial F_0}{\partial t_0})^2} \cdots \frac{1}{(z_n - \frac{\partial F_0}{\partial t_0})^2} \cdot \frac{\partial^2 F_0}{\partial t_0^2} \right].$$

After taking  $\frac{\partial^{n-2}}{\partial t_0^{n-2}}$  on the right-hand side, we get a polynomial in

$$\begin{aligned}
& \frac{j!}{(z_i - \frac{\partial F_0}{\partial t_0})^{j+1}}, \quad i = 1, \dots, n, \quad j \geq 2, \\
& \frac{\partial^k F_0}{\partial t_0^k}, \quad k \geq 2.
\end{aligned}$$

The first few examples are:

$$\begin{aligned}
W_{0,1}(z_1; \mathbf{t}) &= \frac{1}{z_1 - \frac{\partial F_0}{\partial t_0}}, \\
W_{0,2}(z_1, z_2; \mathbf{t}) &= \frac{1}{(z_1 - \frac{\partial F_0}{\partial t_0})^2} \cdot \frac{1}{(z_2 - \frac{\partial F_0}{\partial t_0})^2} \cdot \frac{\partial^2 F_0}{\partial t_0^2},
\end{aligned}$$

- (1)  $\Gamma$  has exactly  $n$  vertices marked by  $\circ$ , and they are all of valence one and are marked by  $z_1, \dots, z_n$  respectively.
- (2)  $\Gamma$  has exactly  $n$  vertices marked by  $\bullet$ , each of which is joined via an edge to a vertex marked by  $\circ$ .

- (3) *There are maybe some vertices of valence  $\geq 3$  marked by  $\circledast$ , and they can only be joined directly to vertices marked by  $\bullet$ .*

*We will refer to the above three kinds of vertices as  $\circ$ -vertices,  $\bullet$ -vertices and  $\circledast$ -vertices respectively. The weight  $w_\Gamma$  is given as usual:*

$$w_\Gamma = \prod_{v \in V(\Gamma)} w_v \cdot \prod_{e \in E(\Gamma)} w_e,$$

where  $w_v$  is given by

$$w_v = \begin{cases} 1, & \text{if } v \text{ is a } \circ\text{-vertex,} \\ \frac{(\text{val}(v)-1)!}{(z_j - \frac{\partial F_0}{\partial t_0})^{\text{val}(v)}}, & \text{if } v \text{ is a } \bullet\text{-vertex joined to a } \circ\text{-vertex marked by } z_j, \\ \frac{\partial^{\text{val}(v)} F_0}{\partial t_0^{\text{val}(v)}}, & \text{if } v \text{ is a } \circledast\text{-vertex,} \end{cases}$$

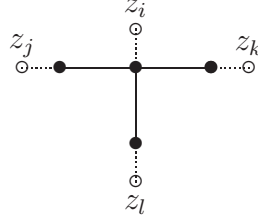
and  $w_e$  is given by:

$$w_e = \begin{cases} 1, & \text{if } e \text{ is incident at a } \circ\text{-vertex or a } \circledast\text{-vertex,} \\ \frac{\partial^2 F_0}{\partial t_0^2}, & \text{if } e \text{ joins two } \bullet\text{-vertices.} \end{cases}$$

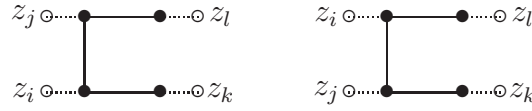
To further illustrate the Feynman rules, let us check them for  $W_{0,4}$ . An application of Leibniz formula gives us:

$$\begin{aligned} W_{0,4}(z_1, \dots, z_4; \mathbf{t}) &= \sum_{i=1}^4 \frac{3!}{(z_i - \frac{\partial F_0}{\partial t_0})^4} \cdot \left( \frac{\partial^2 F_0}{\partial t_0^2} \right)^2 \cdot \prod_{\substack{1 \leq j \leq 4 \\ j \neq i}} \frac{1}{(z_j - \frac{\partial F_0}{\partial t_0})^2} \cdot \frac{\partial^2 F_0}{\partial t_0^2} \\ &+ 2 \sum_{1 \leq i < j \leq 4} \frac{2!}{(z_i - \frac{\partial F_0}{\partial t_0})^3} \cdot \frac{\partial^2 F_0}{\partial t_0^2} \cdot \frac{2!}{(z_j - \frac{\partial F_0}{\partial t_0})^3} \cdot \frac{\partial^2 F_0}{\partial t_0^2} \\ &\cdot \prod_{\substack{1 \leq k \leq 4 \\ k \neq i, j}} \frac{1}{(z_k - \frac{\partial F_0}{\partial t_0})^2} \cdot \frac{\partial^2 F_0}{\partial t_0^2} \\ &+ 3 \sum_{i=1}^4 \frac{2!}{(z_i - \frac{\partial F_0}{\partial t_0})^3} \cdot \frac{\partial^2 F_0}{\partial t_0^2} \cdot \prod_{\substack{1 \leq j \leq 4 \\ j \neq i}} \frac{1}{(z_j - \frac{\partial F_0}{\partial t_0})^2} \cdot \frac{\partial^3 F_0}{\partial t_0^3} \\ &+ \prod_{1 \leq j \leq 4} \frac{1}{(z_j - \frac{\partial F_0}{\partial t_0})^2} \cdot \frac{\partial^4 F_0}{\partial t_0^4}. \end{aligned}$$

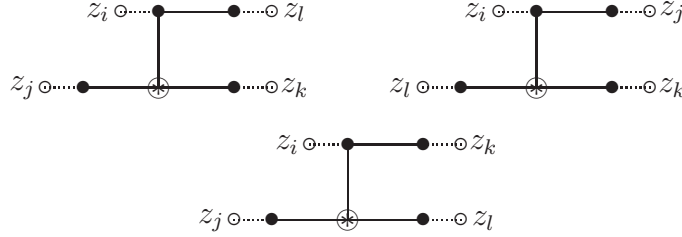
The first summation on the right-hand side corresponds to diagrams of the following form:



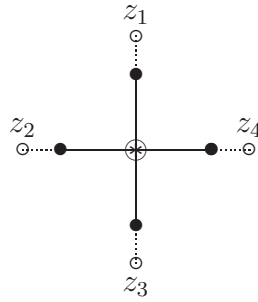
The second summation on the right-hand side corresponds to diagrams of the following form:



Note the appearance of the factor 2 comes from the fact that the above two diagrams have the same contributions. The third summation on the right-hand side corresponds to diagrams of the following form:



Note the appearance of the factor 3 comes from the fact that the above three diagrams all have the same contributions. The last term on the right-hand side corresponds to the diagram:



*Proof of Theorem 9.1.* We use induction on  $n$ . We have seen above that the Theorem holds for  $n = 1$ . Suppose that it holds for some  $n \geq 1$ . Then we apply the loop operator  $B(z_{n+1})$  to get:

$$W_{0,n+1}(z_1, \dots, z_n; \mathbf{t}) = \sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} B(z_{n+1}) w_{\Gamma},$$

where

$$B(z_{n+1})w_\Gamma = B(z_{n+1}) \prod_{v \in V(\Gamma)} w_v \cdot \prod_{e \in E(\Gamma)} w_e.$$

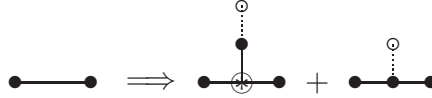
We will write  $B(z_{n+1})w_\Gamma$  as a summation over Feynman diagrams obtained by grafting new branch of the form  $z_{n+1} \circ \cdots \bullet$  to  $\Gamma$  and perhaps at the same time split a  $\otimes$ -vertex. Because  $B(z_{n+1})$  is a derivation, we need to consider  $B(z_{n+1})w_v$  for all  $v \in V(\Gamma)$  and  $B(z_{n+1})w_e$  for all  $e \in E(\Gamma)$ .

Let us consider  $B(z_{n+1})w_e$  first. If  $e$  is incident at a  $\circ$ -vertex or a  $\otimes$ -vertex, then  $w_e = 1$  and so  $B(z_{n+1})w_e = 0$ . This vanishing means there is no grafting on the interior of the edge  $e$ .

If  $e$  joins two  $\bullet$ -vertices, then  $w_e = \frac{\partial^2 F_0}{\partial t_0^2}$ , and so

$$\begin{aligned} B(z_{n+1})w_e &= B(z_{n+1}) \frac{\partial^2 F_0}{\partial t_0^2} = \frac{\partial^2}{\partial t_0^2} B(z_{n+1})F_0 = \frac{\partial^2}{\partial t_0^2} \frac{1}{z_{n+1} - \frac{\partial F_0}{\partial t_0}} \\ &= \frac{1}{(z_{n+1} - \frac{\partial F_0}{\partial t_0})^2} \cdot \frac{\partial^3 F_0}{\partial t_0^3} + \frac{2}{(z_{n+1} - \frac{\partial F_0}{\partial t_0})^3} \cdot \left( \frac{\partial^2 F_0}{\partial t_0^2} \right)^2. \end{aligned}$$

This can graphically represented as follows:



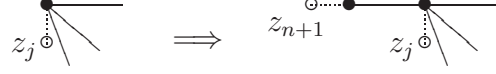
This means there are two ways to graft a branch of the form  $z_{n+1} \circ \cdots \bullet$  to an edge connecting two  $\bullet$ -vertices.

If  $v$  is a  $\circ$ -vertex of  $\Gamma$ , then  $w_v = 1$  and  $B(z_{n+1})w_v = 0$ . This vanishing means that there is no grafting at this vertex. If  $v$  is a  $\bullet$ -vertex of  $\Gamma$ , then

$$\begin{aligned} B(z_{n+1})w_v &= B(z_{n+1}) \frac{(\text{val}(v) - 1)!}{(z_j - \frac{\partial F_0}{\partial t_0})^{\text{val}(v)}} \\ &= \frac{\text{val}(v)!}{(z_j - \frac{\partial F_0}{\partial t_0})^{\text{val}(v)+1}} \cdot \frac{\partial}{\partial t_0} B(z_{n+1})F_0 \\ &= \frac{\text{val}(v)!}{(z_j - \frac{\partial F_0}{\partial t_0})^{\text{val}(v)+1}} \cdot \frac{\partial}{\partial t_0} \frac{1}{z_{n+1} - \frac{\partial F_0}{\partial t_0}} \\ &= \frac{\text{val}(v)!}{(z_j - \frac{\partial F_0}{\partial t_0})^{\text{val}(v)+1}} \cdot \frac{1}{(z_{n+1} - \frac{\partial F_0}{\partial t_0})^2} \cdot \frac{\partial^2 F_0}{\partial t_0^2}. \end{aligned}$$



Pictorially, this can be represented as follows:



If  $v$  is a  $\otimes$ -vertex of  $\Gamma$ , then

$$\begin{aligned} B(z_{n+1})w_v &= B(z_{n+1})\frac{\partial^{\text{val}(v)}F_0}{\partial t_0^{\text{val}(v)}} = \frac{\partial^{\text{val}(v)}}{\partial t_0^{\text{val}(v)}}B(z_{n+1})F_0 \\ &= \frac{\partial^{\text{val}(v)}}{\partial t_0^{\text{val}(v)}}\frac{1}{z_{n+1} - \frac{\partial F_0}{\partial t_0}}. \end{aligned}$$

We will write the derivative  $\frac{\partial^m}{\partial t_0^m}\frac{1}{z_{n+1} - \frac{\partial F_0}{\partial t_0}}$  as a sum over some diagrams.

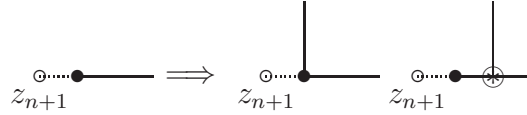
This can be done inductively as follows. For  $m = 0$ , we associate to  $\frac{1}{z_{n+1} - \frac{\partial F_0}{\partial t_0}}$  the diagram  $z_{n+1} \circ \cdots \bullet$ ; note

$$\frac{\partial}{\partial t_0}\frac{1}{z_{n+1} - \frac{\partial F_0}{\partial t_0}} = \frac{1}{(z_{n+1} - \frac{\partial F_0}{\partial t_0})^2} \cdot \frac{\partial^2 F_0}{\partial t_0^2},$$

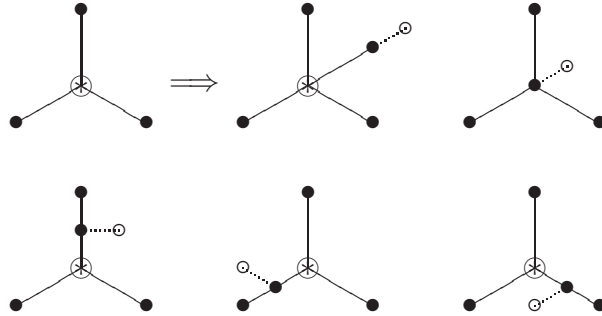
we associate to it the diagram  $z_{n+1} \circ \cdots \bullet \text{---}$ ; i.e., we graft an edge at the  $\bullet$ -vertex of  $z_{n+1} \circ \cdots \bullet$ . Next we note

$$\frac{\partial^2}{\partial t_0^2}\frac{1}{z_{n+1} - \frac{\partial F_0}{\partial t_0}} = \frac{2!}{(z_{n+1} - \frac{\partial F_0}{\partial t_0})^3} \cdot \left(\frac{\partial^2 F_0}{\partial t_0^2}\right)^2 + \frac{1}{(z_{n+1} - \frac{\partial F_0}{\partial t_0})^2} \cdot \frac{\partial^3 F_0}{\partial t_0^3},$$

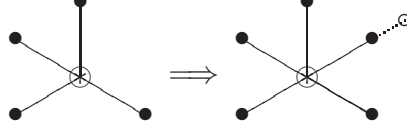
this process can be graphically represented as follows:



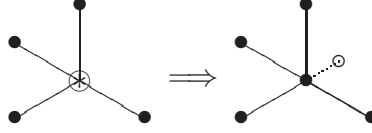
After taking another derivative, we get the following grafting rule for a  $\otimes$ -vertex of valence 3:



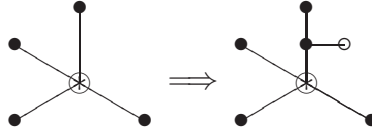
In general, by taking derivatives, one gets the following grafting rules at a  $\otimes$ -vertex. (1) Grafting on the  $\otimes$ -vertex:



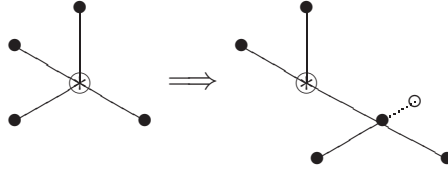
(2) Replacing  $\otimes$ -vertex by  $\bullet$ -vertex:



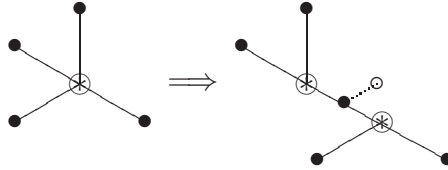
(3) Grafting along an edge:



(4) Splitting of the  $\otimes$ -vertex: There are two kinds of splittings as indicated below:



and



In summary, by induction  $W_{0,n+1}(z_1, \dots, z_n; \mathbf{t})$  can be written as a summation over Feynman diagrams. One also has to take care of the issue of the automorphism groups. That is left to the interested reader.

**9.2. Feynman rules for genus one  $n$ -point functions.** We have shown that

$$(306) \quad F_1 = \frac{1}{2} \log \frac{1}{1 - I_1} = \frac{1}{2} \log \frac{\partial^2 F_0}{\partial t_0^2}.$$

Applying the loop operator  $B(z_1)$ :

$$\begin{aligned} W_{1,1}(z_1; \mathbf{t}) &= B(z_1)F_1 = \frac{1}{2} \left( \frac{\partial^2 F_0}{\partial t_0^2} \right)^{-1} \cdot \frac{\partial^2}{\partial t_0^2} \frac{1}{z_1 - \frac{\partial F_0}{\partial t_0}} \\ &= \frac{1}{2} \frac{1}{(z_1 - \frac{\partial F_0}{\partial t_0})^2} \left( \frac{\partial^2 F_0}{\partial t_0^2} \right)^{-1} \frac{\partial^3 F_0}{\partial t_0^3} + \frac{1}{2} \frac{2!}{(z_1 - \frac{\partial F_0}{\partial t_0})^3} \frac{\partial^2 F_0}{\partial t_0^2}. \end{aligned}$$

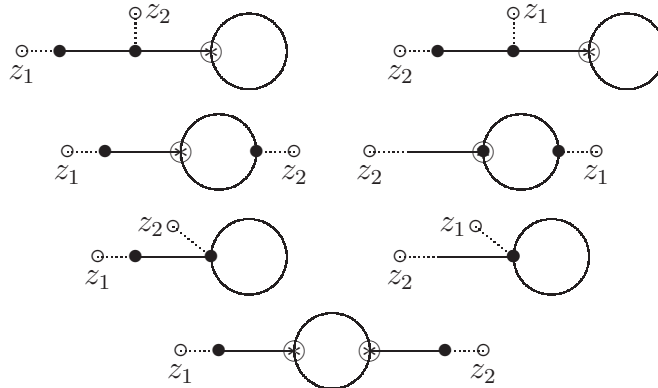
The two terms on the right-hand side correspond to the following two diagrams:



After applying the loop operator  $B(z_2)$  to  $W_{1,1}(z_1; \mathbf{t})$  we get

$$\begin{aligned} W_{1,2}(z_1, z_2; \mathbf{t}) &= 2 \cdot \frac{1}{2} \frac{2!}{(z_1 - \frac{\partial F_0}{\partial t_0})^3} \frac{1}{(z_2 - \frac{\partial F_0}{\partial t_0})^2} \frac{\partial^3 F_0}{\partial t_0^3} \\ &+ 2 \cdot \frac{1}{2} \frac{2!}{(z_2 - \frac{\partial F_0}{\partial t_0})^3} \frac{1}{(z_1 - \frac{\partial F_0}{\partial t_0})^2} \frac{\partial^3 F_0}{\partial t_0^3} \\ &+ \frac{1}{2} \frac{1}{(z_1 - \frac{\partial F_0}{\partial t_0})^2} \frac{3!}{(z_2 - \frac{\partial F_0}{\partial t_0})^4} \left( \frac{\partial^2 F_0}{\partial t_0^2} \right)^2 \\ &+ \frac{1}{2} \frac{1}{(z_2 - \frac{\partial F_0}{\partial t_0})^2} \frac{3!}{(z_1 - \frac{\partial F_0}{\partial t_0})^4} \left( \frac{\partial^2 F_0}{\partial t_0^2} \right)^2 \\ &- \frac{1}{2} \frac{1}{(z_1 - \frac{\partial F_0}{\partial t_0})^2} \frac{1}{(z_2 - \frac{\partial F_0}{\partial t_0})^2} \left( \frac{\partial^3 F_0}{\partial t_0^3} \right)^2. \end{aligned}$$

They correspond to the following diagrams:



By repeatedly applying the loop operators one can write down the Feynman rules for  $W_{1,n}(z_1, \dots, z_n; \mathbf{t})$ .

One can also use (128), (??) etc. to derive Feynman rules for  $W_{g,n}(z_1, \dots, z_n; \mathbf{t})$  ( $g \geq 2$ ).

## 10. SPECTRAL CURVE AND ITS SPECIAL DEFORMATION FOR TOPOLOGICAL 1D GRAVITY

In this Section we show that the genus zero one-point function combined with the gradient of the action function gives rise to the spectral curve and its special deformation for topological 1D gravity. We also establish a uniqueness result for the special deformation.

**10.1. Spectral curve and its special deformation.** We define the spectral curve of the topological 1D gravity by

$$(307) \quad y = \frac{1}{\sqrt{2}} \frac{\partial S(z, \mathbf{t})}{\partial z} + \sqrt{2} W_{0,1}(z; \mathbf{t}),$$

or more concretely:

$$(308) \quad y = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{t_n - \delta_{n,1}}{n!} z^n + \frac{\sqrt{2}}{z} + \sqrt{2} \sum_{n=1}^{\infty} \frac{n!}{z^{n+1}} \frac{\partial F_0}{\partial t_{n-1}}.$$

The reason for the artificial factor of  $\sqrt{2}$  above will be made clear in the next Section. Our computations for  $W_{0,1}$  above yields:

$$(309) \quad y = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{t_n - 1}{n!} z^n + \frac{\sqrt{2}}{z - \frac{\partial F_0}{\partial t_0}},$$

and when restricted to the line  $t_n = 0$  for  $n \geq 1$ ,

$$(310) \quad y = -\frac{1}{\sqrt{2}}(z - t_0) + \frac{\sqrt{2}}{z - t_0}.$$

This is a deformation of the following curve:

$$(311) \quad y = -\frac{1}{\sqrt{2}}z + \frac{\sqrt{2}}{z},$$

which we call the *signed Catalan curve*. Note

$$(312) \quad \frac{z}{\sqrt{2}} = \frac{-y + \sqrt{y^2 + 4}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \binom{2n}{n} y^{-2n-1}.$$

The coefficients of the series on the right-hand side are Catalan numbers  $\frac{1}{n+1} \binom{2n}{n}$  with signs  $(-1)^n$ .

**Theorem 10.1.** *Consider the following series:*

$$(313) \quad y = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{t_n - \delta_{n,1}}{n!} z^n + \frac{\sqrt{2}}{z} + \sqrt{2} \sum_{n=1}^{\infty} \frac{n!}{z^{n+1}} \frac{\partial F_0}{\partial t_{n-1}}.$$

*Then one has:*

$$(314) \quad \frac{1}{2}(y^2)_- = W_{0,1}(z; \mathbf{t})^2.$$

Here for a formal series  $\sum_{n \in \mathbb{Z}} a_n f^n$ ,

$$(315) \quad \left( \sum_{n \in \mathbb{Z}} a_n f^n \right)_+ = \sum_{n \geq 0} a_n f^n, \quad \left( \sum_{n \in \mathbb{Z}} a_n f^n \right)_- = \sum_{n < 0} a_n f^n.$$

*Proof.* This is actually equivalent to the Virasoro constraints for  $F_0$ . Indeed,

$$\begin{aligned} \frac{y^2}{2} &= \left( \frac{1}{2} \sum_{n=0}^{\infty} \frac{t_n - \delta_{n,1}}{n!} z^n + \frac{1}{z} + \sum_{n=1}^{\infty} \frac{n!}{z^{n+1}} \frac{\partial F_0}{\partial t_{n-1}} \right)^2 \\ &= \frac{1}{4} \left( \sum_{n=0}^{\infty} \frac{t_n - \delta_{n,1}}{n!} z^n \right)^2 + \sum_{n=0}^{\infty} \frac{t_n - \delta_{n,1}}{n!} z^n \left( \frac{1}{z} + \sum_{n=1}^{\infty} \frac{n!}{z^{n+1}} \frac{\partial F_0}{\partial t_{n-1}} \right) \\ &\quad + \left( \frac{1}{z} + \sum_{n=1}^{\infty} \frac{n!}{z^{n+1}} \frac{\partial F_0}{\partial t_{n-1}} \right)^2. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{1}{2}(y^2)_- &= \frac{1}{z} + \sum_{n=0}^{\infty} \frac{t_n - \delta_{n,1}}{n!} \frac{\partial F_0}{\partial t_{n-1}} \\ &\quad + \sum_{m \geq 1} \sum_{n=0}^{\infty} \frac{(n+m)!}{n! z^{m+1}} (t_n - \delta_{n,1}) \frac{\partial F_0}{\partial t_{n+m-1}} \\ &\quad + \left( \frac{1}{z} + \sum_{n=1}^{\infty} \frac{n!}{z^{n+1}} \frac{\partial F_0}{\partial t_{n-1}} \right)^2. \end{aligned}$$

The proof is completed by Virasoro constraints for  $F_0$ .  $\square$

**10.2. Uniqueness of special deformation of the Airy curve.** Let us first prove a simple combinatorial result.

**Theorem 10.2.** *There exists a unique series*

$$(316) \quad y = \frac{1}{\sqrt{2}} \sum_{n \geq 0} (v_n - \delta_{n,1}) z^n + \frac{\sqrt{2}}{z} + \sqrt{2} \sum_{n \geq 0} w_n z^{-n-2}$$

such that each  $w_n \in \mathbb{C}[[v_0, v_1, \dots]]$  and

$$(317) \quad (y^2)_- = \left( \frac{1}{z} + \sum_{n \geq 0} w_n z^{-n-2} \right)^2.$$

*Proof.* We begin by rewriting (317) as a sequence of equations:

$$\begin{aligned}
(318) \quad & w_0 = v_0 + v_1 w_0 + v_2 w_1 + v_3 w_2 + \cdots, \\
(319) \quad & w_1 = v_0 w_0 + v_1 w_1 + v_2 w_2 + \cdots, \\
(320) \quad & w_2 = v_0 w_1 + v_1 w_2 + \cdots, \\
(321) \quad & w_3 = v_0 w_2 + \cdots, \\
(322) \quad & \dots\dots\dots
\end{aligned}$$

Write

$$w_n = w_n^{(0)} + w_n^{(1)} + \cdots,$$

where each  $w_n^{(k)}$  consists of monomials in  $v_0, v_1, \dots$  of degree  $k$ . Using such decompositions, one can deduce by induction from the above system of equations:

$$(323) \quad w_n^{(j)} = 0, \quad n \geq 0, \quad j = 0, \dots, n,$$

and furthermore,

$$\begin{aligned}
w_0^{(1)} &= v_0, \\
w_m^{(n)} &= \sum_{j=0}^{\infty} v_j w_{j+m-1}^{n-1}, \quad m \geq 0, \quad n \geq m+2.
\end{aligned}$$

It follows that one can recursively determine all  $w_m^{(n)}$  from the initial value  $w_0^{(1)} = v_0$ .  $\square$

By combining Theorem 10.1 with Theorem 10.2, we then get:

**Theorem 10.3.** *For a series of the form*

$$(324) \quad y = \frac{1}{\sqrt{2}} \frac{\partial S(z, \mathbf{t})}{\partial z} + \sum_{n \geq 0} w_n z^{-n-2},$$

where each  $w_n \in \mathbb{C}[[t_0, t_1, \dots]]$ , the equation

$$(325) \quad (y^2)_- = \left( \frac{1}{z} + \sum_{n \geq 0} w_n z^{-n-2} \right)^2$$

has a unique solution given by:

$$y = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{t_n - \delta_{n,1}}{n!} z^n + \frac{\sqrt{2}}{z} + \sqrt{2} \sum_{n=1}^{\infty} \frac{n!}{z^{n+1}} \frac{\partial F_0}{\partial t_{n-1}}.$$

where  $F_0(\mathbf{u})$  is the free energy of the 1D topological gravity in genus zero.

## 11. QUANTUM DEFORMATION THEORY OF THE SPECTRAL CURVE FOR TOPOLOGICAL 1D GRAVITY

We have already shown that the free energy in genus zero of 1D topological quantum gravity can be used to produce special deformation of its spectral curve. In this Section we will see that this deformation lead to a quantization of the spectral curve that can be used to recover the free energy in all genera.

**11.1. Symplectic reformulation of the special deformation.** One can formally understand  $y$  as a field on the spectral curve. Consider the space consisting of elements of the form:

$$(326) \quad \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{\tilde{u}_n}{n!} z^n + \frac{\sqrt{2}}{z} + \sqrt{2} \sum_{n=1}^{\infty} \tilde{v}_{n-1} \frac{n!}{z^{n+1}}.$$

We regard  $\{\tilde{u}_n, \tilde{v}_n\}$  as linear coordinates on  $V$ , and introduce the following symplectic structure on  $V$ :

$$(327) \quad \omega = \sum_{n=0}^{\infty} d\tilde{u}_n \wedge d\tilde{v}_n.$$

It follows that

$$(328) \quad \tilde{v}_n = \frac{\partial F_0}{\partial u_n}(\mathbf{u})$$

defines a Lagrangian submanifold in  $V$ , and so does

$$(329) \quad \tilde{v}_n = \frac{\partial(\lambda^2 F)}{\partial u_n}(\mathbf{u}).$$

In other words, free energy in all genera produces a deformation of a Lagrangian submanifold.

**11.2. Canonical quantization of the special deformation of spectral curve.** Take the natural polarization that  $\{q_n = \tilde{u}_n\}$  and  $\{p_n = \tilde{v}_n\}$ , one can consider the canonical quantization:

$$(330) \quad \hat{\tilde{u}}_n = \tilde{u}_n, \quad \hat{\tilde{v}}_n = \frac{\partial}{\partial \tilde{u}_n}.$$

Corresponding to the field  $y$ , consider the following fields of operators on the spectral curve:

$$(331) \quad \hat{y} = \sum_{n=0}^{\infty} \beta_{-n-1} z^n + \frac{\beta_0}{z} + \sum_{n=0}^{\infty} \beta_{n+1} z^{-n-2},$$

where the operators  $\beta_m$  are defined by:

$$(332) \quad \beta_{-(k+1)} = \lambda^{-2} \frac{1}{\sqrt{2}} \frac{\tilde{t}_k}{k!}, \quad \beta_{k+1} = \lambda^2 \sqrt{2} (k+1)! \frac{\partial}{\partial t_k}, \quad \beta_0 = \sqrt{2}.$$

**11.3. The bosonic Fock space.** As usual, the operators  $\{\beta_{n+1}\}_{n \geq 0}$  are the annihilators while the operators  $\{\beta_{-(n+1)}\}_{n \geq 0}$  are the creators. Given  $\beta_{n_1+1}, \dots, \beta_{n_k+1}$ , their normally ordered products are defined:

$$(333) \quad : \beta_{n_1+1}, \dots, \beta_{n_k+1} := \beta_{n'_1+1} \cdots \beta_{n'_k+1},$$

where  $n'_1 \geq \dots \geq n'_k$  is a rearrangement of  $n_1, \dots, n_k$ . Denote  $|0\rangle$  the vector 1. The bosonic Fock space  $\Lambda$  is the space spanned by elements of form  $\beta_{-(n_1+1)} \cdots \beta_{-(n_k+1)} |0\rangle$ , where  $n_1, \dots, n_k \geq 0$ . On this space one can define a Hermitian product by setting

$$(334) \quad \langle 0|0\rangle = 1,$$

$$(335) \quad \beta_{n+1}^* = \beta_{-(n+1)}.$$

For a linear operator  $A : \Lambda \rightarrow \Lambda$ , its vacuum expectation value is defined by:

$$(336) \quad \langle A \rangle = \langle 0|A|0\rangle.$$

**11.4. Regularized products of two fields.** We now study the product of the fields  $\hat{y}(z)$  with  $\hat{y}(z)$ . This cannot be defined directly, because for example,

$$(337) \quad \langle 0|\hat{y}(z)\hat{y}(z)|0\rangle = \frac{2}{z^2} + \frac{1}{z^2} \sum_{n \geq 0} (n+1).$$

To fix this problem, we follow the common practice in the physics literature by using the normally ordered products of fields and regularization of the singular terms as follows. First note

$$\begin{aligned} \hat{y}(z) \cdot \hat{y}(w) &= : \hat{y}(z)\hat{y}(w) : + \sum_{n=0}^{\infty} (n+1) z^{-n-2} w^n \\ &= : \hat{y}(z)\hat{y}(w) : + \frac{1}{(z-w)^2}. \end{aligned}$$

It follows that

$$(338) \quad \langle \hat{y}(z) \cdot \hat{y}(w) \rangle = \frac{1}{(z-w)^2},$$

hence

$$(339) \quad \hat{y}(z) \cdot \hat{y}(w) = : \hat{y}(z) \cdot \hat{y}(w) : + \langle \hat{y}(z) \cdot \hat{y}(w) \rangle.$$



Now we have

$$\hat{y}(z + \epsilon) \cdot \hat{y}(z) = : \hat{y}(z + \epsilon) \hat{y}(z) : + \frac{1}{\epsilon^2}.$$

We define the regularized product of  $\hat{y}(z)$  with itself by

$$(340) \quad \hat{y}(z) \odot \hat{y}(z) = \hat{y}(z)^{\odot 2} := \lim_{\epsilon \rightarrow 0} (\hat{y}(z + \epsilon) \hat{y}(z) - \frac{1}{\epsilon^2}) =: \hat{y}(z) \hat{y}(z) : .$$

In other words, we simply remove the term that goes to infinity as  $\epsilon \rightarrow 0$ , and then take the limit.

**11.5. Virasoro constraints and mirror symmetry for 1D topological gravity.** The following result establishes the mirror symmetry of the theory of 1D topological gravity and the quantum deformation theory of its spectral curve:

**Theorem 11.1.** *The partition function  $Z$  of the topological 1D gravity is uniquely specified by the following equation:*

$$(341) \quad (\hat{y}(z)^{\odot 2})_- Z = 0.$$

*Proof.* By the definition of  $\hat{y}(z)$  and (340), one gets:

$$\begin{aligned} \frac{1}{2}(\hat{y}(z)^{\odot 2})_- &= (\beta_0 \beta_{-1} + \sum_{n=1}^{\infty} \beta_{-n-1} \beta_n) z^{-1} + (\sum_{n=0}^{\infty} \beta_{-n-1} \beta_{n+1} + \frac{\beta_0^2}{2}) z^{-2} \\ &+ \sum_{m \geq 1} (\sum_{n=0}^{\infty} \beta_{-(n+1)} \beta_{n+m+1} + \frac{1}{2} \sum_{\substack{j+k=m \\ j,k \geq 0}} \beta_j \beta_k) z^{-m-2}. \end{aligned}$$

It is then straightforward to see that (341) is equivalent to the Virasoro constraints (251)-(254).  $\square$

## 12. CONCLUDING REMARKS

In this paper we have focused mainly on the problems of computing free energy, partition function and  $n$ -point functions of topological 1D gravity. Besides the flow equation and polymer equation that appeared long ago in the literature, we have developed the techniques of changing coordinates to the  $I$ -coordinates.

In the process of computing  $n$ -point functions, the importance of the role played by the loop operator has been made clear. Furthermore, in the study of spectral curve, its special deformation and its quantum deformation theory, it has become clear that the gradient of the effective action function of topological 1D gravity and the loop operator can be combined into a free boson field, and the partition function can

be identified with a vector in the bosonic Fock space uniquely specified by the Virasoro constraints, determined themselves by the special deformation of the spectral curve. This can be compared with an earlier work [18] in which we have done similar things for topological 2D gravity. Such phenomena can be regarded as examples of holography principle applied to the spectral curve. The special deformation of the spectral curve can be detected by taking residues at infinity, and the whole theory produces an element in the Fock space associated to the infinity.

This work provides the foundation for further developments of the theory of topological 1D gravity, its generalizations to include topological matters, and generalizations and comparisons with the theory of topological 2D gravity, etc. We will report on these developments in forthcoming work.

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